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ASYMPTOTIC SOLUTIONS OF MODEL EQUATIONS IN NONLINEAR ACOUSTICS

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CONTENTS

This paper uses the method of matched asymptotic expansions to derive asymptotic solutions to various problems in nonlinear acoustics. Model equations, generalizing the well known Burgers equation to include effects of cylindrical or spherical spreading and of non-equilibrium relaxation, are given and regarded as governing the propagation. Solutions are sought for initial or boundary conditions of N-wave or harmonic wave form. For a thermoviscous medium, the small parameter upon which the asymptotic expansions are predicated is an inverse acoustic Reynolds number; for a relaxing medium it is the product of wave frequency and relaxation time. The complete asymptotic solution for N-waves in a thermoviscous fluid is known, in the case of plane motion, from the Cole–Hopf solution of Burgers's equation. Here a similarly complete solution is found for spherical N-waves, with the exception of one region of space–time in which an irreducible nonlinear problem remains unsolved. In this region the outer limiting behaviour is, nevertheless, determined, so that the solutions in all other regions are completely fixed. For cylindrical N-waves an irreducible problem again results, but the motion can be followed right through into its 'old age' phase aside from an undetermined purely numerical constant. Correct results are obtained here for the 'correction due to diffusivity' to the weak-shock theory prediction of shock centre location for plane, cylindrical and spherical N-waves. These results indicate a non-uniformity in weak shock theory at large times, and also, in the case of spherical N-waves, reveal a large time non-uniformity in the Taylor shock solution. Harmonic waves, plane, cylindrical and spherical, in thermoviscous fluids and relaxing fluids are considered, and the asymptotic solutions are found to leading order in most of the many overlapping asymptotic regions of space–time. A single dimensionless function remains undetermined in the important case of spherical harmonic waves. We have also been unable to find scalings and differential equations describing precisely how a discontinuity is formed at the front of a partly dispersed shock in a relaxing gas, though the shock centre is located for both fully and partly dispersed shocks. The harmonic wave solutions unify and extend certain solutions (the Fay, Fubini and old-age solutions) which are well known in the nonlinear acoustics literature, and the amplitude saturation and scaling laws for the old age regime are in accord with experiments on high amplitude spherical waves in water.

1. Introduction

This paper deals with the asymptotic solution of certain model equations which have found considerable popularity in nonlinear acoustics, and related fields, over the past two decades. Of course, the best known is the Burgers equation, introduced in its familiar form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

into unsteady aerodynamics by Cole (1951), and into nonlinear acoustics by Mendousse (1953) in an equivalent form

$$\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial \theta}=\epsilon\frac{\partial^2 v}{\partial \theta^2}, \tag{1.2}$$

better suited to boundary value problems. Lighthill (1956) first provided the basis for regarding Burgers's equation as a rational approximation to the full equations for plane unidirectional unsteady motion in a non-relaxing thermoviscous fluid, rather than simply a model having certain structural features in common with the full equations. Further detail in this direction was provided by Hayes (1958), while more recently Moran & Shen (1966) and Leibovich & Seebass (1974) – among many others – have shown how Burgers's equation arises in appropriate circumstances from the application of matched expansion and multiple scaling techniques (respectively) to the full equations.

For plane progressive waves, the basic criterion for the validity of Burgers's equation (whether in the initial-value problem form (1.1) with $u(x, t=0)$ prescribed, or in the boundary-value problem form (1.2) with $v(x=0,\theta)$ prescribed) is that changes due to nonlinear convection, to the nonlinearity of the pressure-density relation, and to thermoviscous diffusion should be 'slow' on a wavelength scale. For progressive waves with cylindrical or spherical symmetry a generalized Burgers equation can be derived, in the form

$$\frac{\partial v}{\partial x}+\frac{jv}{2x}-v\frac{\partial v}{\partial \theta}=\epsilon\frac{\partial^2 v}{\partial \theta^2}, \tag{1.3}$$

corresponding to (1.2), *only* when a further 'far field' approximation is made (Lighthill 1956). The far field approximation again ensures that changes – this time associated with the linear geometrical spreading effect – should be slow on a wavelength scale. In (1.3) the integer j is determined by the number of dimensions in which the wave can spread; $j = 1$ gives cylindrical spreading, $j = 2$ spherical spreading.

When $j = 0$, the Cole–Hopf transformation (Cole 1951, Hopf 1950) yields a linearization, and hence exact general solution to (1.1), and there is a corresponding transformation for the form (1.2). The implications of this solution for various initial waveforms of particular physical interest have been explored by many authors, including Lighthill (1956), Hayes (1958), Blackstock (1964) and Whitham (1974). The subtleties concealed by the Cole–Hopf solution have not, however, been exhaustively revealed by these studies, and we shall need to examine certain aspects in more detail yet, in §3 below.

For $j \neq 0$ no linearizing transformation of (1.3) is known. Equation (1.3) is a canonical equation, embodying the conflict between linear propagation, cylindrical or spherical spreading, nonlinear convection and thermoviscous diffusion, and as far as we are aware, only one exact solution has yet been found. That solution is a similarity solution for cylindrical flow, with the form

$$x^{\frac{1}{2}}v(x, \theta) = \Phi(x/\theta^2). \tag{1.4}$$

Its existence was first noted by Chong & Sirovich (1973); the closed-form expression for Φ was given by Rudenko & Soluyan (1977, p. 70); and Sinai (1976) found the profile of the axisymmetric body which would produce the similarity solution in the steady supersonic flow problem governed by an equation of the form (1.3). In the absence of further exact solutions to (1.3) we are severely handicapped; even the numerical solutions of an equation equivalent to (1.3) with an N-wave initial condition given by Sachdev & Seebass (1973) are less useful than they might be. This is because the numerical integration has been stopped at too early a stage, as judged from the calculations to be presented here in §3 at any rate. These asymptotic calculations are really needed *before* any numerical integrations are carried out, in order to establish the typical distances and times over which various processes are significant.

Other approximate solutions of (1.3) have invariably been of an ad hoc nature. Parker (1975), for example, applies the Cole–Hopf transformation

$$v = 2\epsilon \frac{\partial}{\partial \theta} \ln \psi \tag{1.5}$$

to give

$$\frac{\partial \psi}{\partial x} + j \frac{\psi \ln \psi}{2x} = \epsilon \frac{\partial^2 \psi}{\partial \theta^2}, \tag{1.6}$$

and then approximates this equation as

$$\frac{\partial \psi}{\partial x} + j \frac{\psi}{2x} = \epsilon \frac{\partial^2 \psi}{\partial \theta^2} \quad (\psi \approx 1), \tag{1.7}$$

on the basis that v is in some sense 'small'. This is, however, at best a local approximation, and solutions of the resulting linear equation (1.7) have no overlap with other asymptotic solutions of (1.3). For harmonic waves equation (1.3) is equivalent to an infinite set of coupled nonlinear ordinary differential equations for the development with range x of the Fourier amplitudes. Various proposals have been made (Shooter, Muir & Blackstock 1974; Fenlon 1971) for the truncation of this set, and for the derivation of a single equation for the fundamental amplitude. These proposals are motivated by physical reasoning, and have no mathematical justification: though we shall in fact see that the form given by Shooter, Muir & Blackstock is, in several important respects at least, consistent with our asymptotic results.

When we come to relaxing media, the papers best known in the West, giving a model equation of the type

$$\left(1 + \Omega \frac{\partial}{\partial \theta}\right)\left(\frac{\partial v}{\partial x} - v \frac{\partial v}{\partial \theta} - \epsilon \frac{\partial^2 v}{\partial \theta^2}\right) = \Gamma \frac{\partial^2 v}{\partial \theta^2}, \tag{1.8}$$

replacing the Burgers equation, are those by Ockendon & Spence (1969) and Blythe (1969). An equation equivalent to (1.8) had, however, previously been given by Polyakova, Soluyan & Khokhlov (1962). Again, cylindrical or spherical spreading can be included, as in (1.3), and it is easy to derive (1.8) (or an equivalent initial-value problem form) from the full equations in a formal manner by using the multiple scaling technique of Leibovich & Seebass (1974). The validity of (1.8) does not depend upon any high or low value of the parameter Ω, equal to the product of a typical wave frequency with the relaxation time, but does depend upon the assumed *smallness* of the difference between the sound speeds in 'frozen' and 'equilibrium' conditions, the parameter Γ being proportional to this difference. Equation (1.8) shows that, in all parts of a low frequency wave where the relaxation operator $(1+\Omega\,\partial/\partial\theta)$ may be replaced by unity, relaxation effects are equivalent to a bulk viscosity Γ (in dimensionless form) additional to the thermo-viscosity ϵ. Rudenko & Soluyan (1977, p. 93) attempt to improve on this interpretation by expanding $(1+\Omega\,\partial/\partial\theta)^{-1}$ as $(1-\Omega\,\partial/\partial\theta+O(\Omega^2))$ for $\Omega \ll 1$, so that (1.8) becomes

$$\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial \theta}=(\epsilon+\Gamma)\frac{\partial^2 v}{\partial \theta^2}-\Gamma\Omega\frac{\partial^3 v}{\partial \theta^3}, \tag{1.9}$$

a fusion of the Burgers and Korteweg–de Vries equations. It is well known (cf. van Wijngaarden 1972; Whitham 1974, p. 484) that in some parameter ranges the steady shock transition solution to (1.9) may exhibit oscillations on the downstream side, while in other ranges the solution will be of the more normal non-oscillatory type. This behaviour has been observed experimentally in bubbly liquids, for which (1.9) is indeed the correct model equation, and Rudenko & Soluyan claim that the prediction is in accord with observations of oscillatory waveforms in an ordinary relaxing gas (see their fig. 4.5, p. 93). Their claim is, however, invalid for the following reasons (which presuppose, of course, that (1.8) is the correct equation to model the situation of fig. 4.5):

(i) the steady shock transition solution to (1.8) does not exhibit oscillations;

(ii) wherever the final, dispersive, term in (1.9) is significant, (1.9) is not a valid approximation to (1.8);

(iii) even if (1.9) is accepted, its solutions do not exhibit oscillations on the downstream side if ϵ, Γ, Ω are all positive and if $\Gamma\Omega \ll (\epsilon+\Gamma)$.

Our own work on equation (1.8) fails to give any indication of the origin of the observed oscillatory waveforms (except in bubbly liquids), and we are inclined to regard the oscillations discussed by Rudenko & Soluyan as being associated more with the electronics of the experiment than with the dynamics of relaxing gases.

As for exact solutions of (1.8), only one is known, corresponding to the steady progressing shock transition. If $\epsilon = 0$ the solution (discussed by all the authors just cited) can be expressed in the closed form given in §4 below; it represents a continuous fully dispersed relaxing shock for wave strengths below a critical value, whereas for greater wave strengths a partly dispersed shock results, the solution is discontinuous and requires the introduction of thermoviscous effects to provide fine-scale structure in the shock front. High frequency asymptotics for (1.8) are easily obtained; as $\Omega \to \infty$ (1.8) takes the form

$$\frac{\partial q}{\partial x}-q\frac{\partial q}{\partial \zeta}+\lambda q=0, \tag{1.10}$$

the Varley–Rogers (1967) equation, which can be immediately solved by the method of characteristics. Solutions obtained in this way are analysed by Rudenko & Soluyan (1977, p. 95).

Perturbations beyond the leading order solution $q(x, \zeta)$ are also obtained without difficulty of principle. The low frequency limit $\Omega \to 0$ has not been studied systematically, and the work of §§4 and 6 below therefore deals with a matched expansion approach to low frequency problems. Some aspects of these problems have been touched upon before, for example in problems of steady supersonic relaxing gas dynamics treated by Clarke & Sinai (1977) and Sinai & Clarke (1978), also with the aid of matched expansions. There are interesting parallels between this field and that of nonlinear acoustics which deserve to be more widely appreciated; so also do singular perturbation techniques in general among workers in nonlinear acoustics.

Many other model equations have now been derived to characterise nonlinear acoustic propagation in other circumstances – in bubbly liquids, radiating gases and aerosols, in stratified fluids, in tubes with wall boundary layer dispersion and dissipation, in nonlinear beams with diffraction and in gases suffering absorption of intense laser radiation. A review of the model equations for all these situations, and more, is given by Crighton (1979). In future work we hope to extend the present studies in some of these directions. For the present we attempt to elucidate the asymptotic structure of problems involving the propagation of plane, cylindrical and spherical waves, of N-wave or harmonic wave form, in thermoviscous and relaxing fluids. In §2 we outline the basis for the model equations to be studied, and in §§3–6 we analyse these equations in the context of specific initial or boundary values. The concluding section advocates the widespread use of singular perturbation techniques in nonlinear acoustics (particularly in underwater acoustics), while an appendix proves a result for the long-time development of a certain solution to the spherical wave Burgers equation which is crucial to the derivation, in §3, of the 'old age' solution for spherical N-waves. (The reader should note that the symbols used in this Introduction are used in a generic sense, and do not necessarily correspond with the precisely defined symbols used in the sequel.)

2. Basis for the model equations

When finite-amplitude or real gas effects are taken into account, sound propagation no longer satisfies the ordinary wave equation. If these effects are small, however, in some sense to be made more precise later, then the sound wave changes form slowly as it propagates forward. If, in addition, the disturbance is non-planar, having spherical or cylindrical symmetry instead, then the geometry also contributes to the deformation of the waveform. Approximate equations describing this gradual distortion can be obtained by various methods, and we consider below two specific examples.

(i) *Finite-amplitude waves under the influences of viscosity, heat conduction and curved geometry*

The sound wave has plane, cylindrical or spherical symmetry (symbolized by $j = 0, 1, 2$ respectively), and to ensure that deformation of shape is slow we require:

(*a*) $U/a_0 \ll 1$, where U is the velocity of the medium and a_0 is the small-signal sound speed: this means that finite amplitude effects are locally small.

(*b*) $l/r \ll 1$, where l is a typical wavelength of the disturbance and r is the distance to the centre of symmetry: meaning that geometric spreading effects are small.

(*c*) $\varDelta/(a_0 l) \ll 1$, where $\varDelta$ is the 'diffusivity of sound' (Lighthill 1956), i.e. thermoviscous diffusive effects are small.

Since these deformations are slow, the wave equation $U_{tt} = a_0^2 U_{rr}$ applies to a first order, and so a small amplitude initial waveform will split into two, one travelling to the right, the other to the left, at the sound speed a_0. In considering the deformation of the waveform by the above effects we restrict attention to the right-travelling disturbance. Accordingly we introduce the coordinate $X = r - a_0 t$, while t is defined so that the centre of the wave is at $X = 0$; it is immaterial, in virtue of (*b*) above, which part of the wave we choose as centre. Then we refer the reader to Leibovich & Seebass (1974) for derivation of the following equation:

$$\frac{\partial U}{\partial t} + \frac{\gamma+1}{2} U \frac{\partial U}{\partial X} + j \frac{U}{2t} = \tfrac{1}{2}\Delta \frac{\partial^2 U}{\partial X^2}. \tag{2.1}$$

Here γ is the adiabatic exponent, or alternatively $1 + B/A$, where A, B are coefficients in an assumed expansion of pressure in powers of the density generalizing the adiabatic law to media other than gases (see, for example, Blackstock 1972). Equation (2.1) is referred to as a generalized Burgers equation, reducing to the familiar Burgers equation in the case of plane flow, when $j = 0$.

Suppose we start with some initial disturbance $u_0(X)$ at a time t_0 (which could be produced by a piston motion at $r = t_0/a_0$), having length scale l_0 and amplitude U_0. The general character of the subsequent motion is discussed in Leibovich & Seebass (1974). No non-trivial solutions of (2.1) are known unless $j = 0$, with the sole exception of the similarity solution (1.4) for $j = 1$, but fortunately the most interesting results occur in the limit of small diffusion, when $\Delta/U_0 l_0 \ll 1$. In this case convection will cause the wave to steepen and shock in the familiar way, resulting in thin shocks in which a balance with diffusion has been struck. As explained by Leibovich & Seebass, the waveform at this stage has a sawtooth structure. To elucidate the subsequent development of the sawtooth we shall here consider the initial condition

$$U(X, t_0) = \begin{cases} U_0 X/l_0 & \text{for} \quad |X| < l_0, \\ 0 & \text{for} \quad |X| > l_0, \end{cases} \tag{2.2}$$

which contravenes condition (*c*) because of the sharp steps at $X = \pm l_0$, but quickly adjusts itself (in what we will call the 'embryo shock region') to a standard 'N-wave' with thin shocks at either end and for which (*c*) *is* satisfied. So from this time on the model equation will indeed describe the physical problem. In any event, (2.1) and (2.2) are also of interest as a vehicle for an illustration of the treatment of nonlinear wave problems by the method of matched asymptotic expansions (m.a.e.).

To make treatment of (2.1, 2.2) easier we make the transformations

$$\left.\begin{aligned} &V = (t/t_0)^{\frac{1}{2}j} U/U_0, \quad x = X/l_0, \\ &T = \begin{cases} 1 + \frac{1}{2}(\gamma+1)\, U_0 (t - t_0)/l_0 & \text{if} \quad j = 0, \\ 1 + (\gamma+1)\, U_0 (t_0^{\frac{1}{2}} t^{\frac{1}{2}} - t_0)/l_0 & \text{if} \quad j = 1, \\ 1 + \frac{1}{2}(\gamma+1)\, [U_0 t_0/l_0] \ln (t/t_0) & \text{if} \quad j = 2, \end{cases} \end{aligned}\right\} \tag{2.3}$$

and obtain the initial value problem cited at the beginning of § 3, with the dimensionless parameters

$$T_0 = (\gamma+1)\, U_0 t_0 / j l_0 \quad (j \neq 0), \tag{2.4}$$

$$\epsilon = \begin{cases} \Delta/(\gamma+1)\, U_0 l_0 & \text{if} \quad j = 0, \\ 2\Delta/(\gamma+1)\, U_0 l_0 T_0 & \text{if} \quad j = 1, \\ \Delta e^{-1/T_0}/(\gamma+1)\, U_0 l_0 & \text{if} \quad j = 2. \end{cases} \tag{2.5}$$

In the interesting case when geometric and convective effects are of comparable importance, T_0 is of order unity and ϵ is small. Thus in § 3 we keep $T_0 > 0$ fixed and let $\epsilon \to 0+$.

(ii) *Finite-amplitude waves with relaxation and thermoviscous effects*

Here the sound waves will be taken to be planar, and we will require:

(*a*) $U/a_0 \ll 1$,

(*b*) $\varDelta/a_0 l \ll 1$,

(*c*) $(a_\infty - a_0)/a_\infty \ll 1$,

where a_0 and a_∞ are the equilibrium and frozen sound speeds respectively. On this basis Ockendon & Spence (1969) give a derivation of the following equation for unidirectional propagation:

$$\left(1+\tau\frac{\partial}{\partial t}\right)\left\{\frac{\partial U}{\partial t}+\left(\frac{\gamma+1}{2}U+a_\infty\right)\frac{\partial U}{\partial X}-\tfrac{1}{2}\varDelta\frac{\partial^2 U}{\partial X^2}\right\} = (a_\infty - a_0)\frac{\partial U}{\partial X}, \tag{2.6}$$

in which τ is the relaxation time, and the frame of reference is at rest. Now U satisfies

$$\frac{\partial U}{\partial t}+a_0\frac{\partial U}{\partial X} = 0$$

to a first order (as can be seen from (2.6)), and so in the smaller terms we may replace $\partial U/\partial t$ by $-a_0\,\partial U/\partial X$, or vice-versa. Defining $T = t - X/a_0$ as a retarded time based on the zero frequency sound speed, and using the above rule we find that

$$\left(1+\tau\frac{\partial}{\partial T}\right)\left(\frac{\partial U}{\partial X}-\frac{\gamma+1}{2a_0^2}U\frac{\partial U}{\partial T}-\frac{\varDelta}{2a_0^3}\frac{\partial^2 U}{\partial T^2}\right) = \frac{\tau(a_\infty - a_0)}{a_0^2}\frac{\partial^2 U}{\partial T^2}. \tag{2.7}$$

The boundary condition we take in §§4 and 5 is

$$U(0, T) = U_0 \sin \omega T, \tag{2.8}$$

a widely studied condition of fundamental importance. This condition does not quite correspond to that required by a sinusoidal piston motion because the condition is imposed at $X = 0$ rather than on the moving piston face. We disregard this objection on two counts, namely (1) that it is well known that the particular nonlinear effect associated with finite displacement of the piston is a purely local one, of no significance away from the boundary, and (2) that in many cases there may not be any 'piston' present, but one may have instead experimental data at a fixed station $X = 0$ and wish to ask the legitimate question as to how those data will evolve under nonlinear and other effects.

Rescaling variables in (2.7) and (2.8) according to

$$\theta = \omega T, \quad u = U/U_0, \quad \text{and} \quad x = [(\gamma+1)\,U_0\,\omega X/2a_0^2]. \tag{2.9}$$

we obtain

$$\left(1+a\epsilon\frac{\partial}{\partial \theta}\right)\left(\frac{\partial u}{\partial x}-u\frac{\partial u}{\partial \theta}-\delta\frac{\partial^2 u}{\partial \theta^2}\right) = \epsilon\frac{\partial^2 u}{\partial \theta^2}, \tag{2.10}$$

with

$$\left.\begin{aligned} \epsilon &= 2\omega\tau(a_\infty - a_0)/(\gamma+1)\,U_0,\\ a &= \tfrac{1}{2}(\gamma+1)\,U_0/(a_\infty - a_0),\\ \delta &= \varDelta\omega/(\gamma+1)\,U_0\,a_0. \end{aligned}\right\} \tag{2.11}$$

The most interesting case involves $a = O(1)$ and $\epsilon \ll 1$ with very little viscosity (which we include only to smooth off the discontinuity in any partly dispersed relaxation shock). This allows us to pick up the whole range of thin relaxation shocks, both fully dispersed and partly dispersed, for shock heights $O(1)$. The reader is referred to Ockendon & Spence (1969) for a discussion of the relaxation shock structure.

This completes our setting up of the problems we shall consider here; there are of course many other similar problems which can be treated by the same methods. But we go into details only for the equations and boundary conditions discussed above as illustrations of the techniques of m.a.e. and of the undoubted power of these techniques in difficult problems of nonlinear acoustics.

3. N-waves for the generalized Burgers equation

The model equation to describe nonlinear acoustic propagation in a thermoviscous gas can, as explained in §2, be transformed to read

$$\frac{\partial V}{\partial T}+V\frac{\partial V}{\partial x}=\epsilon g(T)\frac{\partial^2 V}{\partial x^2}, \tag{3.1}$$

in which ϵ is an inverse Reynolds number and

$$g(T)=1,\quad \tfrac{1}{2}(T+T_0-1),\quad \exp(T/T_0), \tag{3.2}$$

for motion with plane, cylindrical or spherical symmetry, respectively. We consider the asymptotic solution of (3.1) as $\epsilon\to 0$ for the N-wave problem defined by

$$V(x,1)=\begin{cases} x & \text{for} \quad |x|<1,\\ 0 & \text{for} \quad |x|>1.\end{cases} \tag{3.3}$$

We note that the exact solution to the N-wave problem is antisymmetric about $x=0$, so that we consider only $x>0$. Next note that the $\epsilon=0$ solution is

$$V_0=\begin{cases} x/T & \text{for} \quad x<T^{\frac{1}{2}},\\ 0 & \text{for} \quad x>T^{\frac{1}{2}}.\end{cases} \tag{3.4}$$

Observe at this point that the solution V_0 holds to all algebraic orders in ϵ as the outer solution, i.e.

$$V(x,T)=V_0(x,T)+o(\epsilon^n) \tag{3.5}$$

for all positive n.

This loss-less solution has a discontinuity which must be smoothed off in the usual way by a shock via the scaling

$$x^*=(x-T^{\frac{1}{2}})/\epsilon, \tag{3.6}$$

and matching of an inner solution to (3.4). In the inner region we find

$$V=V_0^*(x^*,T)+o(1), \tag{3.7}$$

where, after straightforward matching to V_0,

$$V_0^*=\tfrac{1}{2}T^{-\frac{1}{2}}\left\{1-\tanh\left[\frac{x^*-A(T)}{4T^{\frac{1}{2}}g(T)}\right]\right\}. \tag{3.8}$$

This is effectively Taylor's (1910) solution for the structure of a steady state thermoviscous shock in which convection and diffusion balance; the shock is thin, and therefore locally plane, so that the function $g(T)$ representing curvature effects is constant through the shock front (Lighthill 1956, p. 323). This state of affairs does not persist to indefinitely large distances, however, as we shall see in due course.

The function $A(T)$, locating the centre of the thermoviscous shock, will now be found by two methods.

(i) To use the integral conservation technique we integrate (3.1) from 0 to $x > T^{\frac{1}{2}}$, to get

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_0^x V\,\mathrm{d}x = -\epsilon g(T)\left\{\frac{\partial V}{\partial x}(0,T)-\frac{\partial V}{\partial x}(x,T)\right\}+\tfrac{1}{2}\{V^2(0,T)-V^2(x,T)\}$$

in which, on the right, we can put

$$\frac{\partial V}{\partial x}(0,T) = T^{-1},\quad \frac{\partial V}{\partial x}(x,T) = 0,\quad \text{and}\quad V(0,T) = V(x,T) = 0,$$

since both points are taken to lie in the outer region, where (3.4) holds with an exponentially small error. Then we integrate over T and apply the initial conditions (3.3) to obtain

$$\int_0^x V\,\mathrm{d}x = \tfrac{1}{2}-\epsilon\int_1^T \frac{g(t)}{t}\,\mathrm{d}t. \tag{3.9}$$

The constant $(\frac{1}{2})$ is the prediction of the 'equal areas' rule of 'weak-shock theory' (see for example, Whitham 1974, ch. 9); the second term in (3.9) is a correction to weak-shock theory arising from diffusive effects, and ultimately dominates the first term at large times.

Lighthill (1956, equation (156)) first derived the first order 'correction due to diffusion' for the shock position in plane flow. Sachdev (1975) attempted to derive comparable results for cylindrical and spherical N-waves using a conservation law technique discussed by Lighthill (1956, p. 334) and elaborated upon by Murray (1968). Sachdev's results are, however, seriously in error, as we shall show not only by the use of the conservation principle but also by the use of matched asymptotic expansions; both Murray (1968) and Sachdev (1975) give the impression that m.a.e. must necessarily fail to locate the shock precisely because of the exponentially smooth matching of the shock with the main wave, but that impression is not upheld by detailed analysis.

To evaluate the integral on the left hand we need more information about the inner solution, so that we continue the inner shock expansion with

$$V = V_0^* + \epsilon V_1^* + o(\epsilon), \tag{3.10}$$

and we define the functions

$$\left.\begin{aligned} f_0(x^*) &= V_0^* + T^{-\frac{1}{2}}\{H(x^*)-1\},\\ f_1(x^*) &= V_1^* + T^{-1}x^*\{H(x^*)-1\},\end{aligned}\right\} \tag{3.11}$$

where H denotes the Heaviside step function. Introduce the expansion operators E_n, E_n^* up to $O(\epsilon^n)$ in the outer (x) and inner (x^*) regions, respectively, and then we have

$$\begin{aligned} E_n(f_0+\epsilon f_1) &= E_n\{V_0^*+\epsilon V_1^* + xT^{-1}[H(x-T^{\frac{1}{2}})-1]\},\\ &= E_n E_1^*(V) - E_1^* E_n(V),\\ &= 0,\end{aligned}$$

by the asymptotic matching principle (Van Dyke 1975, p. 220). But from the form (3.8) we have $E_n(f_0) = 0$, so that we must also have $E_n(f_1) = 0$, from which it follows that f_1 is smaller than any inverse power of x^* as $x^* \to \pm\infty$, and so, that f_1 is integrable over x^*. If we now form an additive composite for V to $O(\epsilon)$ we find that

$$V = V_0 + f_0 + \epsilon f_1 + o(\epsilon)$$

uniformly for $x > 0$. Insert this in (3.9), note that

$$\int_0^x V_0\,\mathrm{d}x = \tfrac{1}{2},\quad \int_0^x f_0\,\mathrm{d}x = \epsilon T^{-\frac{1}{2}}A(T) + o(\epsilon),$$

and that
$$\int_0^x f_1\,dx = \epsilon\int_{-\infty}^{+\infty} f_1\,dx^* + o(\epsilon) = O(\epsilon)$$
because of the integrability of f_1 over x^*, and then we find
$$\epsilon T^{-\frac{1}{2}}A(T) + o(\epsilon) = -\epsilon\int_1^T \frac{g(t)}{t}\,dt,$$
so that
$$A(T) = -T^{\frac{1}{2}}\int_1^T \frac{g(t)}{t}\,dt. \tag{3.12}$$

(ii) To use m.a.e. to locate the shock centre we calculate V_1^* (in **3.10**) explicitly. We have
$$\frac{\partial V_0^*}{\partial T} + \frac{\partial}{\partial x^*}(V_0^*\,V_1^*) - \tfrac{1}{2}T^{-\frac{1}{2}}\frac{\partial V_1^*}{\partial x^*} = g(T)\frac{\partial^2 V_1^*}{\partial x^{*2}},$$
in which we set
$$y = [x^* - A(T)]/4T^{\frac{1}{2}}g(T),$$
and use the known form (3.8) of the function V_0^*; the equation may then be integrated to give
$$\begin{aligned} V_1^* = {} & \frac{dA}{dT} - T^{-\frac{1}{2}}g(T)\,(y^2\operatorname{sech}^2 y + 2y\tanh y - 1 - 2y) + G(T)\,(y\operatorname{sech}^2 y + \tanh y) \\ & + K(T)\operatorname{sech}^2 y + 4T^{\frac{1}{2}}\frac{dg}{dT}\{y - \ln(\cosh y)\tanh y + \tanh y \\ & + \operatorname{sech}^2 y\,[y\,(\ln 2 + \tfrac{1}{2}y) + \tfrac{1}{2}\operatorname{diln}(1 + \exp 2y)]\}. \end{aligned} \tag{3.13}$$
Here diln denotes the dilogarithm,
$$\operatorname{diln}(x) = -\int_1^x \frac{\ln t}{t-1}\,dt \tag{3.14}$$
(see Abramowitz & Stegun 1964, p. 1004 and Lewin 1958), while $G(T)$ and $K(T)$ are as yet arbitrary functions, to be determined by matching. From this we find
$$\begin{aligned} E_1(V_0^* + \epsilon V_1^*) &= V_0 + \epsilon\left\{\frac{dA}{dT} - AT^{-1} + T^{-\frac{1}{2}}g(T) - G(T) - 4T^{\frac{1}{2}}\frac{dg}{dT}(1 + \ln 2)\right\} \quad \text{for} \quad x < T^{\frac{1}{2}}, \\ &= \epsilon\left\{\frac{dA}{dT} + T^{-\frac{1}{2}}g(T) + G(T) + 4T^{\frac{1}{2}}\frac{dg}{dT}(1 + \ln 2)\right\} \quad \text{for} \quad x > T^{\frac{1}{2}}, \end{aligned}$$
and the matching principle allows us to set both coefficients of ϵ to zero. Elimination of $G(T)$ then gives
$$dA/dT + T^{-\frac{1}{2}}g(T) - \tfrac{1}{2}AT^{-1} = 0,$$
so that
$$A(T) = A(1) - T^{\frac{1}{2}}\int_1^T \frac{g(t)}{t}\,dt, \tag{3.15}$$
from which we then find
$$G(T) = -4T^{\frac{1}{2}}(dg/dT)\,(1 + \ln 2) - \tfrac{1}{2}AT^{-1}. \tag{3.16}$$

Expression (3.15) is identical with (3.12) except for the presence of the constant $A(1)$ which must, in the spirit of m.a.e. be evaluated by use of the initial condition. However, the solution we have so far cannot satisfy the initial conditions, since as $T \to 1+$ our inner solution (3.8) tends to a form which is certainly not a sharp step, because $g(1) > 0$. This indicates the need for a new region – the *embryo shock region*, which will occur in all our problems – in which the shock adjusts from a step to the fully developed Taylor form in which convection and diffusion are in a steady state balance. For the N-wave problem it turns out that the appropriate scalings for the embryo shock region are

$$\hat{x} = (x-1)/\epsilon, \quad \hat{T} = (T-1)/\epsilon, \quad V(\hat{x}, \hat{T}, \epsilon) = O(1). \tag{3.17}$$

These scalings are not the same as those used by Lighthill (1956, § 8.3) in his discussion of the same topic (under the title 'region of shock formation') for the plane flow of a thermoviscous gas, using the Cole–Hopf solution of Burgers's equation. The different scalings arise because here the shocks are present at the initial instant, whereas Lighthill considers a general initial value problem in which the shocks are eventually produced by convective steepening. In § 4 we discuss plane harmonic waves in a relaxing gas, and for that problem find precisely the same scales as those of Lighthill for the description of the embryo shock region.

We thus try here

$$V(\hat{x}, \hat{T}, \epsilon) = \hat{V}_0(\hat{x}, \hat{T}) + o(1), \tag{3.18}$$

and find that

$$\frac{\partial \hat{V}_0}{\partial \hat{T}} + \hat{V}_0 \frac{\partial \hat{V}_0}{\partial \hat{x}} = g(1) \frac{\partial^2 \hat{V}_0}{\partial \hat{x}^2}, \tag{3.19}$$

i.e. $\hat{V}_0$ satisfies the ordinary Burgers equation. The initial conditions are

$$\hat{V}_0(\hat{x}, 0) = 1 - H(\hat{x}), \tag{3.20}$$

and the solution can be found from the Cole–Hopf linearizing transformation (Lighthill 1956, p. 299) in the form

$$\hat{V}_0 = \left\{1 + \frac{\operatorname{erfc}[-\hat{x}/(4\hat{T}g(1))^{\frac{1}{2}}] \exp[-(\hat{T}-2\hat{x})/4g(1)]}{\operatorname{erfc}[(\hat{x}-\hat{T})/(4\hat{T}g(1))^{\frac{1}{2}}]}\right\}^{-1}. \tag{3.21}$$

Express $\hat{V}_0$ in terms of the inner shock variables x^* and T and expand for $\epsilon \to 0$, to get

$$\hat{V}_0\left(\hat{x} = x^* + \frac{T^{\frac{1}{2}}-1}{\epsilon}, \quad \hat{T} = \frac{T-1}{\epsilon}\right) = \frac{1}{2}\left\{1 - \tanh\left(\frac{x^*}{4g(1)}\right)\right\} + o(1),$$

which, when matched to the appropriate $(\hat{x}, \hat{T})$ asymptotics of (3.8), yields

$$A(1) = 0,$$

and confirms the equivalence of our two methods in predicting the result (3.12).

The integral conservation technique is much more efficient than m.a.e. as a method for calculating $A(T)$, although, on the other hand, if one wanted to find the second term V_1^* of the shock structure anyway then the two methods are comparable in efficiency. Note also the advantage of m.a.e. in throwing up naturally the need for an 'embryo shock region' which might otherwise have gone unnoticed, and furthermore that, if faced with a different problem for which no suitable conservation law were apparent, then m.a.e. would undoubtedly be the method to use.

To complete the discussion related to $A(T)$, note that the embryo shock solution $\hat{V}_0$ matches the outer solution V_0, which in turn satisfies the initial conditions. We still have to calculate the function $K(T)$, and since we do not wish to find explicitly the next term in the fully developed shock expansion

$$V = V_0^* + \epsilon V_1^* + \epsilon^2 V_2^* + o(\epsilon^2) \tag{3.22}$$

(a wish which must be understandable in view of the complexity of V_1^*), we will use the integral conservation technique. By arguments similar to those used previously we find $E_2(V_2^*) = 0$, so that $V_2^*(x^*) = o(x^{*-2})$ at infinity and is thus integrable. The composite expansion to $O(\epsilon^2)$ is now

$$V = V_0 + f_0 + \epsilon f_1 + \epsilon^2 V_2^* + o(\epsilon^2)$$

uniformly for $x > 0$, which is to be inserted into (3.9). Arguing as before we find

$$\int_{-\infty}^{+\infty} f_1 \, dx^* = 0 = \int_{-\infty}^{+\infty} [V_1^* + T^{-1}x^*\{H(x^*) - 1\}] \, dx^*,$$

from which, if the explicit form of V_1^* and the expression (3.16) for $G(T)$ are used to evaluate the integral, we find

$$K(T) = -T^{-\frac{3}{2}}A^2/[16g(T)]. \tag{3.23}$$

Note that the term involving $K(T)$ in (3.13) represents nothing more than a correction to $A(T)$, i.e. an $O(\epsilon^2)$ correction to the shock location in weak-shock theory.

Explicit evaluation of $A(T)$ in the three cases of physical interest, defined by (3.2), yields

$$A(T) = -T^{\frac{1}{2}} \ln T \quad \text{(plane)}, \tag{3.24a}$$

$$A(T) = -\tfrac{1}{2}T^{\frac{1}{2}}\{T - 1 + (T_0 - 1)\ln T\} \quad \text{(cylindrical)}, \tag{3.24b}$$

$$A(T) = -T^{\frac{1}{2}}\{\mathrm{Ei}(T/T_0) - \mathrm{Ei}(T_0^{-1})\} \quad \text{(spherical)}, \tag{3.24c}$$

where Ei is the exponential integral defined by

$$\mathrm{Ei}\,(x) = \fint_{-\infty}^{x} t^{-1} \exp t \, dt$$

(see Abramowitz & Stegun 1964, p. 228). Formula (3.24*a*) agrees with the result of Lighthill (1958, equation 156) whereas with $T_0 = 1$ our problem coincides with that considered by Sachdev (1975) and our results for cylindrical and spherical geometry differ substantially from his. Sachdev's error in fact lies in following too literally Murray's (1968) version of the integral conservation technique; Murray considers only constant-coefficient equations (though as we have shown here, that is not a necessary feature) and at one stage Sachdev implicitly takes the function $g(T)$ as constant.

At this point it would appear that we have achieved our aim of obtaining an asymptotic solution to the problem posed. However, as is usual in these problems, there is a non-uniformity in our solutions for large T. The basis for the present representation, in terms of an outer loss-less flow plus fully developed Taylor shocks rests upon the assumptions

(i) that the shock width must be small compared with the overall scale, $O(T^{\frac{1}{2}})$, of the N-wave,

(ii) that the 'correction due to diffusivity' must not displace the shock too far from its location according to weak shock theory, and

(iii) that the Taylor steady shock solution (3.8) itself remains valid as a leading order approximation in the variables x^* and T.

These assumptions are violated in a different order at large times depending on whether we consider plane, cylindrical or spherical waves, so that we now have to treat these cases individually.

Plane waves

Here we have $g(T) = O(1)$, $A(T) = O(T^{\frac{1}{2}} \ln T)$ as $T \to \infty$. The shock width in Taylor's solution is $\epsilon T^{\frac{1}{2}} g(T)$ and the shock centre is located at $T^{\frac{1}{2}} + \epsilon A(T)$. We shall regard the term $K(T)\,\mathrm{sech}^2 y$ in (3.13) as incorporated in the argument of the Taylor solution, as it represents a *uniformly* small correction to that argument. Condition (i) is always satisfied, but (ii) is violated, i.e. weak-shock theory fails, when $\ln T = O(\epsilon^{-1})$. We test condition (iii) by examining the ratio $\epsilon V_1^*/V_0^*$ for fixed y as $T \to \infty$, and see at once that (iii) is violated, i.e. Taylor's solution will no longer be valid, at the same time, $\ln T = O(\epsilon^{-1})$, as that at which weak shock theory fails. Thus the non-uniformity still arises in a narrow transition region, outside which the loss-less N-wave solution continues to hold. The difficulty is that the *location* of the transition region is not known accurately for large T; all we know is that for $T = O(1)$ the shock centre is at $T^{\frac{1}{2}} + \epsilon A(T) + O(\epsilon^2)$, and we need in fact to extract from $O(\epsilon^2)$ and all higher terms all the contributions giving rise to the non-uniformity, and then sum them in an appropriate form. This we have been unable to do, because of the great complexity of the terms. A similar difficulty arises also in the formation of a discontinuity in the front of a relaxing shock; see § 4.

In the present case we can, nonetheless, make progress without analysing the 'translational' nonuniformity. Define

$$T_1(\epsilon) = \exp(1/2\epsilon), \quad \boldsymbol{T} = T/T_1; \tag{3.25}$$

the corresponding scalings for x and V are determined as the 'distinguished' scalings (Cole 1968, p. 10) which lead to the least degenerate form of the governing differential equation. We find that

$$\boldsymbol{x} = x/[\epsilon T_1]^{\frac{1}{2}}, \tag{3.26}$$

and

$$V(\boldsymbol{x}, \boldsymbol{T}, \epsilon) = [\epsilon T_1^{-1}]^{\frac{1}{2}}\{V_0(\boldsymbol{x}, \boldsymbol{T}) + o(1)\}, \tag{3.27}$$

where

$$\frac{\partial V_0}{\partial \boldsymbol{T}} + V_0 \frac{\partial V_0}{\partial \boldsymbol{x}} = \frac{\partial^2 V_0}{\partial \boldsymbol{x}^2}. \tag{3.28}$$

Not surprisingly, when the non-uniformity arises (i.e. for $T = O(T_1)$) all terms in the governing equation are of comparable magnitude. We require a solution of (3.28) which matches the Taylor shock solution in the variables (x^*, T) and the loss-less N-wave solution with variables (x, T). Such a solution can be found from the Cole–Hopf transformation as

$$V_0(\boldsymbol{x}, \boldsymbol{T}) = \boldsymbol{x}/\boldsymbol{T}\{\boldsymbol{T}^{\frac{1}{2}} \exp(\boldsymbol{x}^2/\boldsymbol{T}) + 1\}; \tag{3.29}$$

it is noteworthy that this exact solution of the original equation (3.1) does much more than match the previous solutions: it is itself a uniformly valid solution to leading order everywhere except in the embryo shock region (and therefore it does not satisfy the initial condition (3.3)).

The solution (3.29) holds over the whole of the waveform, and there is no reason to suspect a further non-uniformity at larger times, as all three terms in the differential equation have already been called into play in (3.29). Therefore we may let $\boldsymbol{T} \to \infty$, and then we find that (3.29) reduces to a 'dipole' solution of the linear equation

$$\frac{\partial V_0}{\partial \boldsymbol{T}} = \frac{\partial^2 V_0}{\partial \boldsymbol{x}^2}, \tag{3.30}$$

namely

$$V_0 \sim \frac{\boldsymbol{x}}{\boldsymbol{T}^{\frac{3}{2}}} \exp(-\boldsymbol{x}^2/\boldsymbol{T}) = -\frac{1}{2}\frac{\partial}{\partial \boldsymbol{x}}\left\{\frac{1}{\boldsymbol{T}^{\frac{1}{2}}} \exp(-\boldsymbol{x}^2/\boldsymbol{T})\right\}. \tag{3.31}$$

The phase of the motion governed by (3.31) is referred to as 'old-age'.

Cylindrical waves

For this case, $g(T) = O(T)$, $A(T) = O(T^{\frac{3}{4}})$. Whether or not $K(T)\,\mathrm{sech}^2 y$ in V_1^* is regarded as included in the argument of V_0^*, we find that conditions (i), (ii) and (iii) are all violated at the *same* time, $T = O(\epsilon^{-1})$. Thus we define $T' = \epsilon T$, and the appropriate scalings for x and V follow easily from the matching requirements; we set

$$T' = \epsilon T, \quad x' = \epsilon^{\frac{1}{2}} x, \tag{3.32}$$

$$V(x', T', \epsilon) = \epsilon^{\frac{1}{2}} V_0'(x', T') + o(\epsilon^{\frac{1}{2}}), \tag{3.33}$$

and find that the function V_0' satisfies the full generalized Burgers equation for cylindrical waves, in the form

$$\frac{\partial V_0'}{\partial T'} + V_0' \frac{\partial V_0'}{\partial x'} = \tfrac{1}{2} T' \frac{\partial^2 V_0'}{\partial x'^2}. \tag{3.34}$$

Unfortunately, except for the similarity solution mentioned in § 1, which is irrelevant here, there are no known non-trivial solutions of this equation, so that it is impossible to make much further progress. We can, however, note that for $T' \gg 1$ the nonlinear term must presumably become small, so that (3.34) must reduce to the old-age equation

$$\frac{\partial V_0'}{\partial T'} = \tfrac{1}{2} T' \frac{\partial^2 V_0'}{\partial x'^2}, \tag{3.35}$$

whose dipole type of solution is

$$V_0' = -\tfrac{1}{2}C \frac{\partial}{\partial x}\left\{\frac{1}{T'} \exp\left(-\frac{x'^2}{T'^2}\right)\right\},$$

$$= (Cx'/T'^3) \exp(-x'^2/T'^2). \tag{3.36}$$

The constant C is undetermined by this study and its determination remains an important unsolved canonical problem of nonlinear acoustics.

Spherical waves

For the spherical wave it is remarkable that the value of the constant analogous to C in (3.36) can be found exactly, despite our inability to solve the equation analogous to (3.34). This piece of good fortune arises because the first non-uniformity arises in the inner (shock) region alone. We have (with $T_0 = 1$ temporarily)

$$g(T) = e^T, \quad A(T) = O(e^T/T^{\frac{1}{2}}), \quad G = O(T^{\frac{1}{2}}e^T),$$

and

$$K(T) = O(e^T/T^{\frac{5}{2}}),$$

and these show that condition (i) would be violated when $\epsilon e^T \approx 1$, (ii) when $\epsilon T^{-1} e^T \approx 1$ and (iii) when $\epsilon T e^T \approx 1$. It is clear then that (iii) is violated first, and we have a local non-uniformity – a failure of the Taylor solution – in a region thin compared with $T^{\frac{1}{2}}$ and located close to the weak shock location $x = T^{\frac{1}{2}}$.

Define a large time $T_1(\epsilon)$ by

$$\epsilon T_1(\epsilon) \exp(T_1(\epsilon)/T_0) = 1. \tag{3.37}$$

Then examination of the two term inner solution and the basic differential equation suggests the scalings

$$T' = T - T_1, \quad x' = T_1^{\frac{1}{2}}(x - T^{\frac{1}{2}}), \quad V' = VT_1^{\frac{1}{2}} - \tfrac{1}{2}; \tag{3.38}$$

the need for the unusual 'shifting' rather than 'stretching' of the time variable is dictated by the need to balance exponential and algebraic functions of ϵ on the time scale defined by (3.37). This gives the equation

$$\frac{\partial V'}{\partial T'} + V' \frac{\partial V'}{\partial x'} + \frac{1}{2}\left\{1 - \left(\frac{T_1}{T}\right)^{\frac{1}{2}}\right\} \frac{\partial V'}{\partial x'} = e^{T'/T_0} \frac{\partial^2 V'}{\partial x'^2}, \tag{3.39}$$

with

$$\left(\frac{T_1}{T}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\left(\frac{T'}{T_1}\right) + \frac{3}{8}\left(\frac{T'}{T_1}\right)^2 + \dots, \tag{3.40}$$

so that, expanding in the form

$$V'(x', T', \epsilon) = V_0'(x', T') + o(1), \tag{3.41}$$

we obtain

$$\frac{\partial V_0'}{\partial T'} + V_0' \frac{\partial V_0'}{\partial x'} = \exp\left(\frac{T'}{T_0}\right) \frac{\partial^2 V_0'}{\partial x'^2}. \tag{3.42}$$

Again we have to solve the full generalized Burgers equation, and this effectively debars us from finding the complete solution in the (x', T') region if we have a general $O(1)$ value of T_0. (Of course, since we have a two parameter space (ϵ, T_0) we may relate T_0 to ϵ in some prescribed fashion, and in some cases we might then be able to simplify (3.42) to soluble form. All such treatments avoid the central issue posed by the solution of (3.42) with $T_0 = O(1)$, and even though they might be of some practical relevance we shall not consider further any of the many possibilities in which $T_0 \neq O(1)$.)

Now although the solution to (3.42) is not known, it must match the Taylor shock solution according to

$$V_0' \sim -\tfrac{1}{2}\tanh\left[\tfrac{1}{4}x' \exp\left(-T'/T_0\right)\right] \quad \text{as} \quad T' \to -\infty, \tag{3.43}$$

and the main loss-less N-wave according to

$$V_0' \to \mp\tfrac{1}{2} \quad \text{as} \quad x' \to \pm\infty \quad \text{and} \quad T' \to -\infty. \tag{3.44}$$

The crucial point is, however, that even when $T' \gg 1$ the main N-wave solution is still valid, because only condition (iii) has yet been violated. Therefore the solution V_0' must continue to match the loss-less N-wave solution far into the future, so that

$$V_0' \to \mp\tfrac{1}{2} \quad \text{as} \quad x' \to \pm\infty \quad \textit{for all} \quad T'. \tag{3.45}$$

Now we argue that as T' increases, the nonlinear term in (3.42) must decrease faster than the linear terms, so that for large T' we need a solution of the linearized form of (3.42), antisymmetric about $x = 0$ *and satisfying condition* (3.45). There is a unique solution

$$V_0' \sim \tfrac{1}{2}\{\operatorname{erfc}\left[\tfrac{1}{2}x' T_0^{-\frac{1}{2}} e^{-\frac{1}{2}T'/T_0}\right] - 1\}, \tag{3.46}$$

so that, even though we cannot solve (3.42), we can find the long-time asymptotics of V_0'.

The appendix gives a *proof* of the theorem that V_0' has the asymptotic form (3.46) under condition (3.45). It is not, however, evident that a solution V_0' *exists* for the problem defined by (3.42), (3.43), and it is necessary, in the absence of an explicit solution, to prove the existence of V_0' in order to justify the very unusual scalings (3.38). This is a topic taken up elsewhere (Scott 1979).

The 'shock' which takes over from the Taylor form at a time $O(T_1(\epsilon))$ is an evolutionary shock, in which spherical spreading is a controlling mechanism. The asymptotics (3.46) show that as $T' \to +\infty$ the width of this new form of shock is of order $\epsilon^{\frac{1}{2}} \exp\left(\tfrac{1}{2}T/T_0\right)$, comparable with the scale $T^{\frac{1}{2}}$ of the main N-wave when

$$\epsilon T^{-1} \exp\left(T/T_0\right) \approx 1.$$

This defines a time at which a further non-uniformity arises, a gross non-uniformity this time, in which the whole of the wave form must be reexamined. Define

$$\epsilon T_2^{-1}(\epsilon) \exp\left(T_2(\epsilon)/T_0\right) = 1, \tag{3.47}$$

$$\boldsymbol{T} = T - T_2, \quad \boldsymbol{x} = T_2^{-\frac{1}{2}}x, \quad \boldsymbol{V} = T_2^{\frac{1}{2}}V, \tag{3.48}$$

and then we find that $\boldsymbol{V}$ satisfies the equation

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{T}} + T_2^{-1}\boldsymbol{V}\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} = e^{\boldsymbol{T}/T_0}\frac{\partial^2 \boldsymbol{V}}{\partial \boldsymbol{x}^2}, \tag{3.49}$$

so that setting

$$\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{T}, \epsilon) = \boldsymbol{V}_0(\boldsymbol{x}, \boldsymbol{T}) + o(1) \tag{3.50}$$

gives

$$\frac{\partial \boldsymbol{V}_0}{\partial \boldsymbol{T}} = \exp\left(\boldsymbol{T}/T_0\right)\frac{\partial^2 \boldsymbol{V}_0}{\partial \boldsymbol{x}^2}. \tag{3.51}$$

A solution can be found to this linear equation which matches the N-wave and the long time asymptotics (3.46) of the 'shock' solution V_0'. It is

$$V_0 = \tfrac{1}{2}x \operatorname{erfc}\{\nu(x-1)\} - \tfrac{1}{2}x \operatorname{erfc}\{\nu(x+1)\} + (2\pi^{\frac{1}{2}}\nu)^{-1}\exp\{-\nu^2(x+1)^2\} - (2\pi^{\frac{1}{2}}\nu)^{-1}\exp\{-\nu^2(x-1)^2\}, \tag{3.52}$$

in which

$$\nu(T) = \tfrac{1}{2}\{T_0\, e^{T/T_0}\}^{-\frac{1}{2}}. \tag{3.53}$$

As $T\to\infty$ this solution goes over to the dipole form

$$V_0 \sim \frac{4}{3\pi^{\frac{1}{2}}}\, x\nu^3 \exp(-\nu^2 x^2), \tag{3.54}$$

which when written in terms of the original outer variables (x, T) reads

$$V \sim \frac{1}{6\pi^{\frac{1}{2}}}\, T^{\frac{1}{2}}\frac{x}{(\epsilon T_0\, e^{T/T_0})^{\frac{3}{2}}} \exp\left\{-\frac{x^2}{4\epsilon T_0\, e^{T/T_0}}\right\}. \tag{3.55}$$

This solution is expected to remain valid to indefinitely large times T.

This concludes our derivation of asymptotic solutions for N-wave problems in a thermoviscous fluid. Our principal contribution – besides delineating the asymptotic structure for plane, cylindrical and spherical N-waves – has been to find the old age solution for spherical N-waves, an unexpected success in view of our failure to find the solution in the (x', T') region. To completely exhaust the possibilities of the problem posed by (3.1), nothing short of the general exact solution to (3.1) appears to give any chance of improving upon the results we have derived in this section: and even all the recent developments in inverse scattering theory still leave little hope that the general exact solution will be found in the near future. We shall, however, consider the generalized Burgers equation again in §5, this time for the important case of time-harmonic spherical waves, for which the matched expansion approach continues to yield valuable information.

4. Sinusoidal waves in a relaxing, slightly viscous gas

In this section we shall look at a boundary value problem in which a sinusoidal velocity is maintained at $x = 0$, and a plane nonlinear wave propagates away in the positive x-direction into a relaxing thermoviscous gas. For this problem we have, as explained in §2, the model representation

$$(1 + a\epsilon\, \partial/\partial\theta)\,(u_x - uu_\theta - \delta u_{\theta\theta}) = \epsilon u_{\theta\theta}, \tag{4.1}$$

with

$$u(0,\theta) = \sin\theta, \tag{4.2}$$

in which we shall seek asymptotics, as $\epsilon\to 0$, uniformly in range x and in the phase variable θ of linear acoustics. The parameter ϵ is basically a measure of the product of a signal frequency with the time scale of the relaxation process, so that we are looking at a low frequency situation. Alternatively, for $a = O(1)$, ϵ can be regarded as a dimensionless bulk viscosity, while δ is the similarly non-dimensional parameter incorporating equilibrium diffusive effects such as those associated with shear viscosity and thermal conductivity. In order to highlight effects associated with relaxation we shall take $\delta \ll \epsilon$, and specifically $\delta = o(\epsilon^3)$ in order to delay for as long as possible (in terms of expansion in powers of ϵ) the intrusion of thermoviscous forces (which becomes inevitable in some circumstances).

The differential equation preserves the periodicity initiated in the boundary condition (though except in the degenerate Burgers equation case $a = \delta = 0$, or $\epsilon = 0$ it does *not* preserve anti-symmetry about $\theta = 0$). We work in the fundamental period $-\pi \leqslant \theta \leqslant +\pi$ and begin with the outer solution, for x, $\theta = O(1)$, in the form

$$u(x,\theta,\epsilon) = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + o(\epsilon^3), \tag{4.3}$$

with

$$u_0 = \sin p, \tag{4.4}$$

and

$$u_1 = -x \sin p/(1-x\cos p)^2,$$

and where $p(x,\theta)$ defined by

$$p = \theta + x\sin p \tag{4.5}$$

is the exact nonlinear characteristic variable. The solution u_0 can be expressed in Fourier series form (see, for example, Blackstock 1972), as

$$u_0(x,\theta) = 2\sum_{n=1}^{\infty} \frac{J_n(nx)}{nx}\sin n\theta,$$

in which form it is known as the Fubini solution. p is uniquely defined up to the shock formation range $x = 1$, beyond which it becomes multiple-valued. It is, however, single valued at $\theta = \pm\pi$ for all x, and can be defined in a single valued fashion by continuous variation from $\theta = \pm\pi$ to the two sides of $\theta = 0$. But then for $x > 1$ the single valued $p(x,\theta)$ so defined is discontinuous at $\theta = 0$, and as usual we have to insert a shock, with the scaling $\theta^* = \theta/\epsilon$ and the expansion

$$u(x,\theta^*,\epsilon) = u_0^*(x,\theta^*) + \epsilon u_1^*(x,\theta^*) + o(\epsilon). \tag{4.6}$$

Solving for u_0^* and matching to the loss-less outer flow gives

$$a\ln\left(1-\frac{u_0^{*2}}{h^2}\right) + \frac{1}{h}\ln\left(\frac{h-u_0^*}{h+u_0^*}\right) = \{\theta_0(x) - \theta^*\}, \tag{4.7}$$

with the derivatives

$$u_{0\theta^*}^* = \tfrac{1}{2}(h^2 - u_0^{*2})/(1+au_0^*), \tag{4.8}$$

and

$$u_{0x}^* = g(x)\,u_0^* - \tfrac{1}{2}(h^2-u_0^{*2})\left\{\frac{g(x)}{h}\ln\left(\frac{h-u_0^*}{h+u_0^*}\right) + \frac{\mathrm{d}\theta_0}{\mathrm{d}x}\right\}\Big/(1+au_0^*). \tag{4.9}$$

Thus u_0^* is implicitly determined, $\theta_0(x)$ is an as yet undetermined function giving the shock location, and $g(x) = \mathrm{d}\ln h(x)/\mathrm{d}x = \cos p_0/(1-x\cos p_0)$ where $h(x) = \sin p_0$ and $p_0(x)$ is chosen to make $p_0 = x\sin p_0$, $0 \leqslant p_0 \leqslant \pi$.

If $ah < 1$ the solution (4.7) is perfectly satisfactory, the condition $ah < 1$ being the well known condition (Lighthill 1956, p. 343; Ockendon & Spence 1969) for the shock to be 'fully dispersed'. If, on the other hand, $ah \geqslant 1$, the solution is itself multiple valued, and matches the loss-less flow only as $\theta^* \to +\infty$. A discontinuity must be inserted within the 'partly dispersed' relaxing shock, to adjust the velocity to the value, $-h(x)$, which it should have as $\theta^* \to -\infty$.

Examination of (4.7) indicates that the appropriate discontinuous solution has the form

$$u_0^* = -h(x) \quad \text{for} \quad \theta^* < \theta_1(x,\epsilon), \tag{4.10}$$

while for $\theta^* > \theta_1$

u_0^* = the branch of (4.7) obtained by continuous variation from $\theta^* = +\infty$, at which $u_0^* = +h(x)$.

Here θ_1 is defined by

$$u(x,\theta_1,\epsilon) = -a^{-1} \quad \text{when} \quad ah > 1, \tag{4.11}$$

and u refers to the *exact* solution of the full equation (4.1).

The discontinuity introduced in this way into a partially dispersed relaxing shock must, of course, have an internal Taylor type of structure in which thermoviscous effects are significant. To describe this 'subshock' we bring in

$$\bar{\theta} = (\theta^* - \theta_1)/(\delta\epsilon^{-1}), \tag{4.12}$$

and write $$u(x, \bar{\theta}, \epsilon) = \bar{u}_0(x, \bar{\theta}) + o(1).$$

Solving for $\bar{u}_0$, and matching to the θ^* region, for which (4.10) holds, yields

$$\bar{u}_0 = \bar{h}a^{-1} \tanh \{\tfrac{1}{2}\bar{h}a^{-1}[\bar{\theta} - \theta_2(x)]\} - a^{-1}, \tag{4.13}$$

with $\bar{h} = ha - 1$, and where we must have $\theta_2(x) = 0$ in order that the Taylor shock centre coincides with the (unknown) location $\theta^* = \theta_1$ of the discontinuity. For matching to the θ^* region to be completed we need

$$u_0^* \to h - 2a^{-1} \quad \text{as} \quad \theta^* \to \theta_1(x, 0),$$

and thus $$\theta_1(x, 0) = \theta_0 + 2a \ln(\tfrac{1}{2}ah) - h^{-1}\bar{h} \ln \bar{h}. \tag{4.14}$$

We now continue by determining $\theta_0(x)$, using the integral conservation law technique. Integration of the original equation, and use of the boundary conditions and the periodicity of u, produces

$$\int_{-\pi}^{+\pi} u \, d\theta = 0. \tag{4.15}$$

In the integrand here we insert the composite expansion (uniformly valid in $-\pi \leqslant \theta \leqslant +\pi$, except, when $ah > 1$, inside the thermoviscous 'subshock')

$$u = u_0 + \epsilon u_1 - h(x) \operatorname{sgn} \theta + u_0^* + \epsilon f_1 + o(\epsilon), \tag{4.16}$$

with $$f_1(x, \theta^*) = u_1^* + h_1(x) \operatorname{sgn} \theta^* - \theta^* g(x), \tag{4.17}$$

and $$h_1(x) = x \sin p_0/(1 - x \cos p_0)^2. \tag{4.18}$$

If there is a subshock, it makes a contribution $o(\delta)$ to (4.15), while u_0 and u_1 both integrate to zero over a period. Further, f_1 is integrable over $(-\infty, +\infty)$ (as will be proved in a moment) and therefore

$$\int_{-\pi}^{+\pi} \epsilon f_1 \, d\theta = \epsilon^2 \int_{-\infty}^{+\infty} f_1 \, d\theta^* + o(\epsilon^2),$$

so that the integral conservation law gives

$$\begin{aligned}\int_{-\pi}^{+\pi} u \, d\theta &= \int_{-\pi}^{+\pi} (u_0^* - h \operatorname{sgn} \theta) \, d\theta + o(\epsilon), \\ &= \epsilon \int_{-\infty}^{+\infty} (u_0^* - h \operatorname{sgn} \theta^*) \, d\theta^* + o(\epsilon), \\ &= 0,\end{aligned}$$

and so $$\int_{-\infty}^{+\infty} (u_0^* - h(x) \operatorname{sgn} \theta^*) \, d\theta^* = o(1). \tag{4.19}$$

By using (4.8) the integration can be performed, and we find the following:

(i) *Fully dispersed shock, $ah < 1$*

$$\begin{aligned}&\int_{-h}^{+h} \frac{2(u_0^* - h \operatorname{sgn} \theta^*)\,(u_0^* a + 1)}{(h^2 - u_0^{*2})} \, du_0^*, \\ &\qquad = -4ah + 4ah \ln(2h) - 2h\{a \ln(h^2 - v^2) + h^{-1} \ln[(h+v)/(h-v)]\}, \\ &\qquad = 0,\end{aligned}$$

where v denotes $u_0^*(\theta^* = 0)$, and after using (4.7) this gives

$$\theta_0(x) = 2a(\ln 2 - 1). \tag{4.20}$$

Thus, to this order, the 'centre' (i.e. the location at which $u_0^* = 0$) of the fully dispersed shock remains fixed in the waveform, but is shifted from its position $\theta^* = 0$ in a thermoviscous fluid to the point $\theta^* = \theta_0 (< 0)$.

(ii) *Partly dispersed shock, $ah \geqslant 1$*

By looking at the graph of u_0^* against θ^* we find that if (4.19) is to hold, then $\theta_1(x, 0) < 0$, and

$$\begin{aligned}\int_{-\infty}^{+\infty} (u_0^* - h \operatorname{sgn} \theta^*)\, \mathrm{d}\theta^* &= \int_{h-2a^{-1}}^{h} 2\frac{(u_0^* - h \operatorname{sgn}\theta^*)\,(u_0^* a + 1)}{(h^2 - u_0^{*2})}\, \mathrm{d}u_0^* + o(1),\\ &= -4 - 2\ln(ah) + 2ah \ln(4h/a) - 2h\{a \ln(h^2 - v^2)\\ &\quad - h^{-1} \ln[(h-v)/(h+v)]\} + o(1),\end{aligned}$$

again with $v = u_0^*(\theta^* = 0)$. With the aid of (4.7) this gives

$$\theta_0(x) = 2a(\ln 2 - 1) + h^{-1}\{2\bar{h} - (1 + ah)\ln(ah)\}, \tag{4.21}$$

and thus to leading order the location of the discontinuity is given by

$$\theta_1(x, 0) = h^{-1}\{-2 - \bar{h}\ln(\bar{h}/ha)\}. \tag{4.22}$$

In this case the shock centre does not, even to leading order, remain fixed in the waveform.

To demonstrate the integrability of the function f_1, which was crucial to the derivation of (4.19), we argue as in § 3, using the expansion operators E_n, E_n^* to $O(\epsilon^n)$ in terms of fixed values of θ, θ^*. For $x > 1$ it is found that although the terms u_2, u_3 in the outer expansion are discontinuous at $\theta = 0$ they both have limits as $\theta \to 0\pm$. Hence

$$\begin{aligned}E_1^* E_3 u &\equiv E_1^*(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3),\\ &= (h - \epsilon h_1)\operatorname{sgn}\theta + \epsilon\theta^* g,\end{aligned}$$

where h_1 is defined by (4.18) and $g = \mathrm{d}\ln h(x)/\mathrm{d}x$ again. Then defining f_1 by (4.17) we have

$$\begin{aligned}\epsilon E_2(f_1) = E_3(\epsilon f_1) &= E_3(\epsilon u_1^* + \epsilon h_1 \operatorname{sgn}\theta^* - \epsilon\theta^* g),\\ &= E_3 E_1^* u - E_3 u_0^* + \epsilon h_1 \operatorname{sgn}\theta - \theta g,\\ &= E_3 E_1^* u + (\epsilon h_1 - h)\operatorname{sgn}\theta - \epsilon\theta^* g,\\ &= (E_3 E_1^* - E_1^* E_3)\,u,\\ &= 0,\end{aligned}$$

by the asymptotic matching principle. This shows that

$$E_2 f_1(\theta^*) = 0,$$

and hence that $$f_1 = o(\theta^{*-2}) \quad \text{as} \quad |\theta^*| \to \infty,$$

so that f_1 is integrable over $(-\infty, +\infty)$, as required.

It is possible to find another term in the θ^* expansion, describing the structure of the fully or partly dispersed shock. This term should aid us to uncover the mechanism by which the multi-valued first order shock solution u_0^* is developed when $ah > 1$, though we have so far failed to

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pinpoint the mechanism precisely. We therefore address ourselves now to the problem of determining u_1^*, (in $u(x, \theta^*, \epsilon) = u_0^* + \epsilon u_1^* + o(\epsilon)$), which satisfies

$$\left(1 + a\frac{\partial}{\partial\theta^*}\right)\left\{\frac{\partial u_0^*}{\partial x} - \frac{\partial}{\partial\theta^*}(u_0^* u_1^*)\right\} = \frac{\partial^2 u_1^*}{\partial\theta^{*2}}, \tag{4.23}$$

a linear equation, of course, but one whose coefficients involve u_0^*, which is not even known explicitly. An explicit solution for u_1^* can, nonetheless, be found, as follows. Set $v = u_1^* - \theta^* g(x)$, and for a while drop the suffix and asterisk on u_0^*, θ^*. Then

$$\left(1 + a\frac{\partial}{\partial\theta}\right)\left(\frac{\partial u}{\partial x} - \frac{\partial}{\partial\theta}(uv) - g\frac{\partial}{\partial\theta}(\theta u)\right) = \frac{\partial^2 v}{\partial\theta^2}. \tag{4.24}$$

If $ah > 1$, then for $\theta < \theta_1$ the solution for v is simply

$$v = B\,\mathrm{e}^{-h\theta/\bar{h}} + C,$$

where, by virtue of our proof that $f_1 = o(\theta^{-2})$, we must have

$$B = 0 \quad \text{and} \quad C = h_1.$$

Thus

$$u_1^*(x, \theta^*) = \theta^* g(x) + h_1(x), \tag{4.25}$$

for the case $ah > 1$, $\theta^* < \theta_1$.

If, on the other hand, we have $ah < 1$, or $\theta^* > \theta_1$ when $ah > 1$, then (4.7, 4.8, 4.9) apply and we can use

$$\begin{aligned}\int u_x\,\mathrm{d}\theta &= \int 2u_x(ua+1)\,(h^2-u^2)^{-1}\,\mathrm{d}u,\\ &= g\{-2au + ah\ln[(h+u)/(h-u)] + h^{-1}[(h+u)\ln(h+u)\\ &\quad + (h-u)\ln(h-u)] - \ln(h^2-u^2)\} - u\,\mathrm{d}\theta_0/\mathrm{d}x\end{aligned}$$

(again with asterisk and suffix 0 suppressed). This gives

$$\partial[(ua+1)\,v]/\partial\theta + uv = f(u) + G(x), \tag{4.26}$$

where $G(x)$ is unknown, as yet, and

$$\begin{aligned}f(u) = g\Big\{a\ln(h+u)\left[h+u+\frac{a(h^2-u^2)}{2(ua+1)}\right] - a\ln(h-u)\left[h-u-\frac{a(h^2-u^2)}{2(ua+1)}\right] - (\theta_0 + a(2+2\ln h))\,u\Big\}\\ -\tfrac{1}{2}a\left[\frac{\mathrm{d}\theta_0}{\mathrm{d}x} + (\theta_0 + 2a\ln h)\,g\right]\frac{(h^2-u^2)}{(ua+1)} - u\frac{\mathrm{d}\theta_0}{\mathrm{d}x}.\end{aligned} \tag{4.27}$$

Now $f(u) \to hg\,(2a\ln 2 - \theta_0 - 2a) - h\,\mathrm{d}\theta_0/\mathrm{d}x$ as $u \to h$, so that if we assume

$$G(x) \neq -hh_1 - hg(2a\ln 2 - \theta_0 - 2a) + h\,\mathrm{d}\theta_0/\mathrm{d}x$$

then we find, from (4.26), that as $v \to -h_1$ and $u \to h$

$$\frac{\partial v}{\partial\theta} \to \text{a non-zero constant},$$

which contradicts $v \to -h_1$. Thus we need

$$\begin{aligned}G(x) &= -hh_1 - hg(2a\ln 2 - \theta_0 - 2a) + h\,\mathrm{d}\theta_0/\mathrm{d}x,\\ &= -hh_1 + g(\bar{h} - ah\ln ah) + h\,\mathrm{d}\theta_0/\mathrm{d}x,\end{aligned} \tag{4.28}$$

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and the solution for u_1^* is completed by setting

$$w = (ua+1)\,v/(h^2-u^2) \tag{4.29}$$

in (4.26), to give $\partial w/\partial u = 2\{f(u)+G(x)\}\,(ua+1)/(h^2-u^2)^2.$

With the value of f given in (4.27) this can be integrated, at the expense of considerable effort, with the result that

$$w = gah^{-1}\left\{\tfrac{1}{4}h^{-1}\left[\ln\left(\frac{h+u}{h-u}\right)\right]^2+\frac{(ha+1)}{h-u}\ln(h+u)-\frac{(ha-1)}{h+u}\ln(h-u)\right\}+\tfrac{1}{2}Gh^{-3}\ln\left(\frac{h+u}{h-u}\right)$$
$$+\left\{G(x)\,(a+uh^{-2})-(au+1)\frac{\mathrm{d}\theta_0}{\mathrm{d}x}-g(x)\,[\theta_0+2a(1+\ln h)]\,(au+1)\right\}\Big/(h^2-u^2)+K(x), \tag{4.30}$$

$K(x)$ being another unknown function which we now proceed to find using the integral conservation law technique. First we set

$$f_2(\theta^*) = u_2^*-h_2-\theta^*\frac{\partial u_1}{\partial\theta}(x,0)-\frac{\theta^{*2}}{2}\frac{\partial^2 u_0}{\partial\theta^2}(x,0+)\operatorname{sgn}\theta^*,$$

where
$$h_2(x,\theta^*)\begin{cases}= u_2(x,0+) & \text{if}\quad \theta^*>0,\\ = u_2(x,0-) & \text{if}\quad \theta^*<0.\end{cases} \tag{4.31}$$

(In this formula, and subsequently, the asterisk and subscript have now been reinstated.) Then, as before, the matching principle implies that $f_2 = o(\theta^{*-2})$ as $\theta^*\to\pm\infty$. The reasoning leading to this is a little more subtle than previously; suppose that $\delta = \epsilon^5$, and then the next term in the outer expansion is $\epsilon^4 u_4$, where u_4 has a limit as θ tends to zero from either side. Also, from (4.29) and (4.30), v tends to a limit exponentially fast as $\theta^*\to\pm\infty$. These observations allow the matching argument to go through as before, and lead to $f_2 = o(\theta^{*-2})$. Now we remark that u_2^*, u_0, u_1, u_2 – and hence f_2 – remain unchanged when $\delta\neq\epsilon^5$, and thus f_2 has the asserted property anyway.

We use this fact, and the fact that $f_1\to 0$ exponentially fast as $\theta^*\to\pm\infty$, to show, by using a composite expansion to $o(\epsilon^2)$ in the conservation law (4.15), that

$$\int_{-\infty}^{+\infty} f_1\,\mathrm{d}\theta^*+\frac{1}{\epsilon}\int_{-\infty}^{+\infty}(u_0^*-h\operatorname{sgn}\theta^*)\,\mathrm{d}\theta^*+\int_{-\pi}^{+\pi}u_2\,\mathrm{d}\theta = o(1). \tag{4.32}$$

All terms here are known except for u_2, which we write as

$$u_2 = U+V,$$

where
$$\frac{\partial U}{\partial x}-\frac{\partial}{\partial\theta}(u_0 U)-u_1\frac{\partial u_1}{\partial\theta} = \frac{\partial^2 u_1}{\partial\theta^2}, \tag{4.33}$$

$$\frac{\partial V}{\partial x}-\frac{\partial}{\partial\theta}(u_0 V)+a\frac{\partial^3 u_0}{\partial\theta^3} = 0, \tag{4.34}$$

and $U(0,\theta) = V(0,\theta) = 0$. The equation for U preserves antisymmetry about $\theta = 0$, while the equation for V preserves symmetry, and thus we only need the solution for V in order to implement (4.32). That solution is, in fact,

$$V = \tfrac{1}{2}ax(x^2\cos p+2\cos p-3x)/(1-x\cos p)^4,$$

and hence
$$\int_{-\pi}^{+\pi}u_2\,\mathrm{d}\theta = 2\int_0^{\pi}V\,\mathrm{d}\theta = ax\int_0^{\pi}\frac{(x^2\cos p+2\cos p-3x)}{(1-x\cos p)^3}\,\mathrm{d}p,$$
$$= -ah_1/(2-x\cos p_0), \tag{4.35}$$

this last result being very tedious to obtain.

(i) *Fully dispersed shock, $ah < 1$*

We have $\int_{-\infty}^{+\infty} (u_0^* - h\,\mathrm{sgn}\,\theta^*)\,\mathrm{d}\theta^* = 0$, and

$$\int_{-\infty}^{+\infty} f_1\,\mathrm{d}\theta^* = 2\int_{-h}^{+h}\left(w + \frac{h_1(u_0^* a+1)\,\mathrm{sgn}\,\theta^*}{(h^2-u_0^{*2})}\right)\mathrm{d}u_0^*.$$

Inserting the value $w(u_0^*)$ given by (4.30) we find

$$\int_{-\infty}^{+\infty} f_1\,\mathrm{d}\theta^* = -2gah^{-1}[(\ln 2)^2 - 2J] + 4Kh - 4h_1 a,$$

where
$$J = \int_0^1 \frac{\ln(1-x)}{1+x}\,\mathrm{d}x = \tfrac{1}{2}[(\ln 2)^2 - \tfrac{1}{6}\pi^2]$$

(cf. Lewin 1958, equation (1.16)), and this yields

$$K(x) = \tfrac{1}{2}ah^{-1}\{\tfrac{1}{6}gh^{-1}\pi^2 + 3h_1 - \tfrac{1}{2}h_1 x\cos p_0\}. \tag{4.36}$$

(ii) *Partly dispersed shock, $ah \geqslant 1$*

Expand $\theta_1(x,\epsilon)$ in the form

$$\theta_1(x,\epsilon) = \theta_{10}(x) + \epsilon\theta_{11}(x) + o(\epsilon). \tag{4.37}$$

Then
$$\int_{-\infty}^{+\infty} (u_0^* - h\,\mathrm{sgn}\,\theta^*)\,\mathrm{d}\theta^* = -2\epsilon(h-a^{-1})\,\theta_{11} + o(\epsilon),$$

and the correction θ_{11} to the discontinuity location can be found by solving for the next 'sub-shock' term,

$$u(x,\bar\theta,\epsilon) = \bar u_0(x,\bar\theta) + \epsilon\bar u_1(x,\bar\theta) + o(\epsilon). \tag{4.38}$$

The term $\bar u_1$ satisfies
$$a\frac{\partial}{\partial\bar\theta}\left[-\frac{d\theta_{10}}{dx}\frac{\partial\bar u_0}{\partial\bar\theta} - \frac{\partial}{\partial\bar\theta}(\bar u_0\,\bar u_1) - \frac{\partial^2\bar u_1}{\partial\bar\theta^2}\right] = \frac{\partial^2\bar u_1}{\partial\bar\theta^2},$$

and is given explicitly in terms of $y = \tfrac{1}{2}\bar\theta\bar h(x)$ by

$$\bar u_1 = -\mathrm{d}\theta_{10}/\mathrm{d}x + A(y^2\,\mathrm{sech}^2 y - 1 + 2y\tanh y) + B(\tanh y + y\,\mathrm{sech}^2 y) + C\,\mathrm{sech}^2 y, \tag{4.39}$$

A, B, C being unknown functions of x. Matching of the subshock to the surrounding partly dispersed relaxing shock gives

$$A = 0,$$

$$B = -\frac{\mathrm{d}\theta_{10}}{\mathrm{d}x} - h_1 - \theta_{10}g,$$

$$\theta_{11} = -2w(\theta^* = \theta_{10}) - a^2(\mathrm{d}\theta_{10}/\mathrm{d}x + g\theta_{10} + \tfrac{1}{2}h_1),$$

and, by the definition (4.11) of $\theta_1(x,\epsilon)$,

$$C = \mathrm{d}\theta_{10}/\mathrm{d}x. \tag{4.40}$$

This determines the middle term of (4.32), while the third term is given by (4.35). For the first term we have

$$\int_{-\infty}^{+\infty} f_1\,\mathrm{d}\theta^* = \int_{-\infty}^{+\infty}\{u_1^* + h_1\,\mathrm{sgn}\,\theta^* - g\theta^*\}\,\mathrm{d}\theta^*,$$

$$= \int_{\theta_{10}}^{\infty}(u_1^* + h_1 - g\theta^*)\,\mathrm{d}\theta^* + 2h_1\theta_{10} + o(1),$$

$$= 2\int_{h-2a^{-1}}^{h}\left\{w + h_1\frac{(u^*a+1)}{h^2-u^2}\right\}\mathrm{d}u^* + 2h_1\theta_{10} + o(1),$$

and the integral here can be evaluated with use of (4.30) to yield

$$\int_{-\infty}^{+\infty} f_1\, \mathrm{d}\theta^* = 2\ln(1-a^{-1}h^{-1})\,[h^{-1}h_1 - 2a^2g] - gah^{-1}[\ln(1-a^{-1}h^{-1})]^2 + 4a^{-1}w(\theta^* = \theta_{10}) + a\{h_1(1-2h^{-1}a^{-1}) - 2gh^{-1}[\operatorname{diln}(1-a^{-1}h^{-1}) + a^{-1}h^{-1}\bar{h}]\} + o(1), \tag{4.41}$$

where diln (x) is the dilogarithm (see p. 110). Using these results in (4.32), we find

$$K(x) = -\tfrac{1}{2}h^{-1}\{gh^{-2}\bar{h}\ln(1-a^{-1}h^{-1}) - h_1 h^{-1}(\ln ah + 2ah + 1) - gah^{-1}\operatorname{diln}(1-a^{-1}h^{-1}) + \tfrac{1}{2}[\ln(ah)]^2\} + gh^{-2}a\bar{h}\ln(ah) + \tfrac{1}{2}ah_1 x\cos p_0 \tag{4.42}$$

and $$\theta_{11} = h^{-1}\{\ln(1-a^{-1}h^{-1})\,(h^{-1}h_1 - ga^2) - \tfrac{1}{2}gah^{-1}[\ln(1-a^{-1}h^{-1})]^2 - a[h_1(1+a^{-1}h^{-1}) + gh^{-1}(\operatorname{diln}(1-a^{-1}h^{-1}) - 1)] + \tfrac{1}{2}h_1 ax\cos p_0\}. \tag{4.43}$$

We close this discussion of the relaxing shock with the remark that we have analysed the structure of fully and partly dispersed shocks and Taylor subshocks at fixed values of x for which we may have $ah < 1$ or $ah \geqslant 1$. If $a > 1$ there will be two ranges, x_1 and x_2, for which $ah = 1$, and for $x_1 < x < x_2$ the relaxing shock will contain a subshock of the kind analysed here. Around $x = x_1, x_2$, however, there is some further asymptotic structure which we have not been able to elucidate; as x increases through x_1 the relaxing shock structure changes from being smooth (for $x < x_1$) to having a nondifferentiable kink (when $x = x_1$) to being double valued when $x > x_2$. We have not been able to discover either (*a*) a simplified version of (4.1) which describes this evolution or (*b*) a set of scaled variables which implies that (4.1) itself – and nothing simpler – provides the required description. We hope to return to this problem in future work.

Now we consider ranges x larger than the $O(1)$ values dealt with so far, and specifically, the transition into old age. For large x,

$$\left.\begin{aligned} h(x) &\sim \pi(x+1)^{-1} + O(x^{-3}), \\ \text{and}\qquad p(x,\theta) &\sim \pi - (\pi-\theta)/(x+1) + O(x^{-3}), \end{aligned}\right\} \tag{4.44}$$

uniformly in $\theta > 0$. It follows that

$$u_0^* \sim (\pi/x)\tanh\{\tfrac{1}{2}\pi(\theta^* - \theta_0)/x\} + O(x^{-2}), \tag{4.45}$$

also uniformly in θ^*, which indicates a shock thickness ϵx (with respect to θ) and an interference between the shock and the loss-less portion of the wave when $x = O(\epsilon^{-1})$. At this range u_0 and u_0^* both have magnitude $O(\epsilon)$, so that we define

$$x' = \epsilon x, \quad \theta' = \theta, \quad u' = \epsilon^{-1}u = u_0' + o(1), \tag{4.46}$$

there being no need for a rescaling with respect to θ as the rapid changes in that variable have now disappeared. These scalings give

$$\frac{\partial u_0'}{\partial x'} - u_0'\frac{\partial u_0'}{\partial \theta'} = \frac{\partial^2 u_0'}{\partial \theta'^2}, \tag{4.47}$$

and a solution of this Burgers equation (in which relaxation effects are equivalent to a bulk viscosity) is required, matching the shock and loss-less solutions. It was shown by Crighton (1975) how one can do this in a constructive fashion, the basic idea being that the solution of (4.47) should also match a composite expansion, which can be regarded as a single expansion, good for $x = O(1)$ uniformly in θ. The zeroth order composite expansion (of the additive kind) is

$$u_c = u_0 + u_0^* - h(x)\operatorname{sgn}\theta,$$

which in the variables (x', θ') is

$$u_c(x = x'/\epsilon, \quad \theta = \theta', \quad \theta^* = \theta'/\epsilon) = (\epsilon/x)\{\pi \tanh(\tfrac{1}{2}\pi\theta'/x') - \theta'\} + O(\epsilon^2),$$

so that the matching rule is interpreted as implying

$$\epsilon u_0'(\epsilon x, \theta) = (1/x)\{\pi \tanh(\tfrac{1}{2}\pi\theta/\epsilon x) - \theta\} + O(\epsilon). \tag{4.48}$$

As we know that u_0' must be periodic in θ', we can write

$$u_0' = \sum_{n=1}^{\infty} \{A_n(x')\cos(n\theta') + B_n(x')\sin(n\theta')\} + A_0(x'),$$

and therefore Fourier analysis of (4.48) gives (for $x = O(1)$, uniformly in n)

$$\begin{aligned}
\epsilon A_n(\epsilon x) &= o(1),\\
\epsilon B_n(\epsilon x) &= \frac{2}{x}\left\{\frac{1}{n} + 2n \sum_{p=1}^{\infty} \frac{(-1)^p\,(1 - e^{-p\pi^2/\epsilon x}\cos(n\pi))}{(n^2 + \pi^2 p^2/\epsilon^2 x^2)}\right\} + O(\epsilon),\\
&= \frac{2}{x}\left\{\frac{1}{n} + 2n \sum_{p=1}^{\infty} \frac{(-1)^p}{(n^2 + \pi^2 p^2/\epsilon^2 x^2)}\right\} + O(\epsilon),\\
&= 2\epsilon \operatorname{cosech}(n\epsilon x) + O(\epsilon).
\end{aligned}$$

Reconstituting (4.48) from these requirements gives

$$u_0'(x, \theta) = 2\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{\sinh(n\epsilon x)} + O(1) \tag{4.49}$$

for $x = O(1)$ uniformly in θ. Note that the series has magnitude $O(\epsilon^{-1})$ in this region, and so forms the dominant term in (4.49).

Now it is a remarkable fact that the functions given explicitly on the right sides of (4.48) and (4.49) are *each* exact solutions of the Burgers equation, i.e.

$$V_1(x', \theta') = (1/x')\{\pi \tanh(\tfrac{1}{2}\pi\theta'/x') - \theta'\} \tag{4.50}$$

and

$$V_2(x', \theta') = 2\sum_{n=1}^{\infty} \frac{\sin(n\theta')}{\sinh(nx')} \tag{4.51}$$

both satisfy

$$\frac{\partial V}{\partial x'} - V\frac{\partial V}{\partial \theta'} = \frac{\partial^2 V}{\partial \theta'^2}.$$

The proof for V_1 follows by direct substitution; for V_2 it is necessary to use an identity for theta functions in the manner first shown by Cole (1951). Note that V_1 and V_2 are not identical; they differ by $O(1)$ terms when $x' = O(1)$, and by exponentially small terms when $x' \ll 1$. V_1 is not periodic, and cannot be continued periodically in a continuous fashion, whereas V_2 is periodic. We therefore choose

$$u_0'(x', \theta') = V_2(x', \theta'), \tag{4.51 bis}$$

and then (4.49) is trivially satisfied. This solution is known (with some license as to precise correspondence) as the Fay (1931) solution in the nonlinear acoustics literature.

In the case of a thermoviscous fluid the Fay solution holds out to indefinitely large x' (though, of course, it is *not* the solution of Burgers's equation complying with the initial condition), and for large x' it takes the old-age form

$$V_2(x', \theta') \sim 4\,e^{-x'}\sin\theta'.$$

For the relaxing fluid this is not quite the case. We have

$$u_0'(x', \theta') \sim 4\,e^{-x'}\sin\theta' + o(e^{-x'}),$$

which shows that the linearized version of (4.1) must hold at very large ranges. The general solution of that equation with the fundamental time variation is

$$u \sim C \exp\left[-x'(\delta\epsilon^{-1} + (1+a^2\epsilon^2)^{-1})\right] \sin\{\theta' + a\epsilon x'/(1+a^2\epsilon^2)\},$$

which can be matched to the Fay solution to give $C = 4\epsilon$. Thus, uniformly for all θ and $x' \gg 1$ we have the old-age solution as

$$u \sim 4\epsilon \exp\left[-x'(\delta\epsilon^{-1} + (1+a^2\epsilon^2)^{-1})\right] \sin\{\theta + a\epsilon x'/(1+a^2\epsilon^2)\}. \tag{4.52}$$

We now return to an issue we have side-stepped, namely the non-uniformities which arise when a shock is formed. As noted before, we are unable as yet to describe the non-uniformity in the region around $ah = 1$, where a fully dispersed shock becomes a partly dispersed shock. What we are concerned with here is the 'embryo shock region', providing a transition from the single valued loss-less solution (4.4) for $x < 1$ to the fully developed relaxing shock in $x > 1$. We see that u_0 and ϵu_1 are comparable when $x-1 = O(\epsilon^{\frac{1}{2}})$, $\theta = O(\epsilon^{\frac{3}{4}})$, suggesting the scaling

$$x - 1 = \epsilon^{\frac{1}{2}}\hat{x}, \quad \theta = \epsilon^{\frac{3}{4}}\hat{\theta}. \tag{4.53}$$

In this scaling $\epsilon u_1/u_0 = (\hat{x} - \frac{1}{2}\hat{p}^2)^{-2} + o(1)$ and $p = \epsilon^{\frac{1}{4}}\hat{p} + o(\epsilon^{\frac{1}{4}})$, where $\hat{p}$ is defined as a root of

$$\hat{p}^3 - 6\hat{x}\hat{p} - 6\hat{\theta} = 0; \tag{4.54}$$

the highest root for $\hat{\theta} > 0$ and the lowest for $\hat{\theta} < 0$. Then $u_0 = \epsilon^{\frac{1}{4}}\hat{p} + o(\epsilon^{\frac{1}{4}})$, suggesting an expansion

$$u(\hat{x}, \hat{\theta}, \epsilon) = \epsilon^{\frac{1}{4}}\hat{u}_0(\hat{x}, \hat{\theta}) + o(\epsilon^{\frac{1}{4}}), \tag{4.55}$$

where
$$\frac{\partial \hat{u}_0}{\partial \hat{x}} - u_0 \frac{\partial \hat{u}_0}{\partial \hat{\theta}} = \frac{\partial^2 \hat{u}_0}{\partial \hat{\theta}^2}.$$

Making the Cole–Hopf transformation

$$\hat{u}_0 = 2\,\partial \ln H(\hat{x}, \hat{\theta})/\partial\hat{\theta}$$

gives
$$\frac{\partial H}{\partial \hat{x}} = \frac{\partial^2 H}{\partial \hat{\theta}^2},$$

for which we attempt to represent the solution in the form

$$H = \int_{-\infty}^{+\infty} K(q) \exp\{\tfrac{1}{2}q\hat{\theta} + \tfrac{1}{4}q^2\hat{x}\}\,\mathrm{d}q, \tag{4.56}$$

with $K(q)$ to be determined by matching.

For matching to the main wave for $x < 1$ we find

$$\epsilon^{\frac{1}{4}}\hat{u}_0((x-1)/\epsilon^{\frac{1}{2}}, \theta/\epsilon^{\frac{3}{4}}) = \epsilon^{\frac{1}{4}}\hat{p} + O(\epsilon)$$

for $(x-1)$ strictly $O(1)$, uniformly in θ. In particular, we wish to take $x = 1 - D_1$, $\theta = \epsilon^{\frac{1}{4}}D_2$ with D_1, D_2 constants and $D_1 > 0$, for then

$$H\left(\frac{x-1}{\epsilon^{\frac{1}{2}}}, \frac{\theta}{\epsilon^{\frac{3}{4}}}\right) = \int_{-\infty}^{+\infty} K(q) \exp\left\{\frac{1}{2\epsilon^{\frac{1}{2}}}(D_2 q - \tfrac{1}{2}D_1 q^2)\right\}\mathrm{d}q,$$

which can be estimated by the method of steepest descent to yield

$$\tfrac{1}{4}\epsilon\hat{u}_0\left(\frac{x-1}{\epsilon^{\frac{1}{2}}}, \frac{\theta}{\epsilon^{\frac{3}{4}}}\right) = \epsilon^{\frac{1}{4}}\frac{D_2}{D_1} + 2\epsilon^{\frac{3}{4}}\frac{K'(D_2/D_1)}{D_1 K(D_2/D_1)} + O(\epsilon).$$

Also
$$\epsilon^{\frac{1}{4}}\hat{p}\left(\frac{x-1}{\epsilon^{\frac{1}{2}}},\frac{\theta}{\epsilon^{\frac{3}{4}}}\right) = \epsilon^{\frac{1}{4}}\frac{D_2}{D_1}-\epsilon^{\frac{3}{4}}\frac{D_2^3}{6D_1^4}+O(\epsilon),$$

so that the matching rule gives
$$\left.\begin{aligned} K'(q) &= -\tfrac{1}{12}q^3K(q),\\ \text{and}\qquad K(q) &= \exp\left(-\tfrac{1}{48}q^4\right),\end{aligned}\right\} \tag{4.57}$$

apart from a multiplicative constant which does not affect the result for $\hat{u}_0$.

We now show that this solution matches both the shock and the main wave for $x > 1$. We have
$$H = \int_{-\infty}^{+\infty}\exp\{-\tfrac{1}{2}\left(\tfrac{1}{24}q^4-q\hat{\theta}-\tfrac{1}{2}q^2\hat{x}\right)\}\,\mathrm{d}q$$

and for (x, θ) in the main wave the saddle-point of greatest importance occurs at $q = \hat{p}$, leading to a steepest descent estimate
$$\epsilon^{\frac{1}{4}}\hat{u}_0\left(\frac{x-1}{\epsilon^{\frac{1}{2}}},\frac{\theta}{\epsilon^{\frac{3}{4}}}\right) = \epsilon^{\frac{1}{4}}\hat{p}\left(\frac{x-1}{\epsilon^{\frac{1}{2}}},\frac{\theta}{\epsilon^{\frac{3}{4}}}\right)+O(\epsilon).$$

This is exactly the rule for matching to the main wave. In terms of the shock variables x, θ^* we have
$$H\left(\frac{x-1}{\epsilon^{\frac{1}{2}}},\epsilon^{\frac{1}{4}}\theta^*\right) = \epsilon^{-\frac{1}{4}}\int_{-\infty}^{+\infty}\exp\left(\tfrac{1}{2}r\theta^*\right)\exp\left\{-\frac{1}{2\epsilon}\left(\tfrac{1}{24}r^4-\tfrac{1}{2}(x-1)\,r^2\right)\right\}\mathrm{d}r,$$

where there are now three saddle-points, at $r=0$, $r=\pm[6(x-1)]^{\frac{1}{2}}$. Of these the first is irrelevant to the estimate of $\hat{u}_0$, while the contributions from the other two are comparable, and lead to
$$\epsilon^{\frac{1}{4}}\hat{u}_0((x-1)/\epsilon^{\frac{1}{2}},\epsilon^{\frac{1}{4}}\theta^*) = \{6(x-1)\}^{\frac{1}{2}}\tanh\{\tfrac{1}{2}\theta^*[6(x-1)]^{\frac{1}{2}}\}+O(\epsilon),$$

which is again the matching rule between the embryo shock and fully developed shock regions.

We have thus shown that
$$\hat{u}_0(\hat{x},\hat{\theta}) = 2\frac{\partial}{\partial\hat{\theta}}\ln\int_{-\infty}^{+\infty}\exp\left[-\tfrac{1}{48}q^4+\tfrac{1}{4}q^2\hat{x}+\tfrac{1}{2}q\hat{\theta}\right]\mathrm{d}q, \tag{4.58}$$

for which series expansions in $\hat{x}$ or $\hat{\theta}$ can be obtained if desired. Such expansions were in fact used by Lighthill (1956, in calculating the profile in his figure 13), and indeed the scalings (4.53) and the form (4.58) are indicated in that article, where they were ascertained from the exact Cole–Hopf solution to the Burgers equation governing the motion of a thermoviscous fluid. Our work here has essentially been to show how the matching rules lead in a constructive way to these scalings and solutions. This completes our study of harmonic waves in a relaxing fluid.

5. Saturation of cylindrical and spherical harmonic waves with the generalized Burgers equation

It is now convenient to combine the ideas of the previous two sections, and to deal with the very important problem of cylindrical and spherical harmonic waves in a thermoviscous fluid. Let a sinusoidal velocity be prescribed at radial location $r = r_0$ in the form
$$U(r_0,\tau) = U_0\sin(\omega\tau), \tag{5.1}$$

where $\tau = t-(r-r_0)/a_0$ is the retarded time. Then with the approximations outlined in §2 the appropriate model equation takes the form
$$\frac{\partial U}{\partial r}-\frac{\gamma+1}{2a_0^2}U\frac{\partial U}{\partial\tau}+\frac{jU}{2r} = \frac{\varDelta}{2a_0^3}\frac{\partial^2U}{\partial\tau^2}. \tag{5.2}$$

The plane wave case ($j = 0$) has effectively been dealt with, in essence at any rate, in §4, so that here we consider $j = 1, 2$. We make the transformations and definitions

$$\left.\begin{aligned} V &= (r/r_0)^{\frac{1}{2}j} U/U_0, \quad \theta = \omega\tau, \\ R &= 1+R_0\{(r/r_0)^{\frac{1}{2}}-1\} \quad \text{if} \quad j=1, \\ R &= 1+R_0 \ln (r/r_0) \quad \text{if} \quad j=2, \\ g(R) &= \tfrac{1}{2}(R+R_0-1) \quad \text{if} \quad j=1, \\ g(R) &= \exp[(R-1)/R_0]) \quad \text{if} \quad j=2, \\ \epsilon &= 2\omega\Delta/(\gamma+1)\, U_0 R_0 a_0 \quad \text{if} \quad j=1, \\ \epsilon &= \omega\Delta/(\gamma+1)\, U_0 a_0 \quad \text{if} \quad j=2, \end{aligned}\right\} \tag{5.3}$$

and $$R_0 = (\gamma+1)\, U_0 \omega r_0/ja_0^2,$$

to reduce (5.2) and (5.1) to the form

$$\frac{\partial V}{\partial R} - V\frac{\partial V}{\partial \theta} = \epsilon g(R)\frac{\partial^2 V}{\partial \theta^2}, \tag{5.4}$$

$$V(1,\theta) = \sin\theta.$$

In §3 we kept the parameter T_0 fixed as $\epsilon \to 0$; this kept geometric and finite amplitude effects comparable for $T = O(1)$. In the present problem we could do the same thing, keeping R_0 fixed and letting $\epsilon \to 0$. The methods developed in §§3 and 4 are found to be equally applicable here, though we do not give details. We quote only the old-age solution for spherical waves, which is found to be

$$U(r,\tau) = -\frac{4a_0^2 \exp(-\alpha r)\sin(\omega\tau)}{(\gamma+1)\,\omega r \ln\epsilon}, \tag{5.5}$$

where $$\alpha = \tfrac{1}{2}\Delta\omega^2/a_0^3 \tag{5.6}$$

is the usual small-signal attenuation coefficient. The comparable result for the plane wave case is

$$U(x,\tau) = \frac{4\omega\Delta \exp(-\alpha x)\sin(\omega\tau)}{(\gamma+1)\,a_0}, \tag{5.7}$$

displaying the familiar phenomenon of amplitude saturation: the non-dependence of the old-age solution on the source amplitude U_0. By contrast, the spherical wave solution (5.5) displays a 'supersaturation' feature, with the old-age amplitude actually decreasing as U_0 increases!

The situation $R_0 = O(1)$, $\epsilon \to 0$ which gives rise to supersaturation does not, however, correspond to the physical process of increasing the source amplitude U_0 while keeping all other variables fixed. For that situation we need to solve problem (5.4) subject to

$$R_0 = a\epsilon^{-\frac{1}{2}j} \quad \text{as} \quad \epsilon \to 0+, \tag{5.8}$$

where $a > 0$ is fixed. Of course, this has the effect of making geometric spreading changes small for $R = O(1)$, though as we shall see, these effects become important at larger ranges.

The problem is easily solved for $R = O(1)$; in the outer (loss-less) region we have the solution given in (4.4), while in the Taylor shock region we have the expected profile

$$V_0^* = h(R)\tanh[h(R)\,\theta^*/b], \tag{5.9}$$

with $$\theta^* = \theta/\epsilon^{\frac{1}{2}} \quad \text{and} \quad b = a \quad \text{if} \quad j = 1, \tag{5.10}$$

while $$\theta^* = \theta/\epsilon \quad \text{and} \quad b = 2 \quad \text{if} \quad j = 2.$$

Whether $j = 1$ or 2, however, we find now that a breakdown of the (loss-less flow and Taylor shock) solution occurs in both inner and outer regions simultaneously, at a distance given by $R\epsilon^{\frac{1}{2}j} \sim 1$, so that we introduce

$$R' = R\epsilon^{\frac{1}{2}j}, \quad \theta' = \theta, \quad V\epsilon^{-\frac{1}{2}j} = V_0' + o(1), \tag{5.11}$$

to get $$\frac{\partial V_0'}{\partial R'} - V_0' \frac{\partial V_0'}{\partial \theta'} = G(R') \frac{\partial^2 V_0'}{\partial \theta'^2}, \tag{5.12}$$

where $$\left.\begin{aligned} G(R') &= \tfrac{1}{2}(R' + a) \quad \text{for} \quad j = 1, \\ G(R') &= \exp(R'/a) \quad \text{for} \quad j = 2. \end{aligned}\right\} \tag{5.13}$$

The solution $V_0'(R', \theta')$ has to match the solutions in the loss-less and Taylor shock regions. At this stage we have come to an irreducible problem, dependent upon the parameter a. No solution is known in analytic form, so that we cannot progress further, except to note that as $R' \rightarrow \infty$ the solution $V_0'(R', \theta')$ must tend to the old-age form, a sinusoidal solution of the linearized version of (5.12). Thus we anticipate that as $R' \rightarrow \infty$

$$\left.\begin{aligned} V_0' &\sim C_1(a) \exp\{-\tfrac{1}{4}R'^2 - \tfrac{1}{2}aR'\} \sin\theta \quad \text{if} \quad j = 1, \\ V_0' &\sim C_2(a) \exp\{-a \exp(R'/a)\} \sin\theta \quad \text{if} \quad j = 2, \end{aligned}\right\} \tag{5.14}$$

and just as in §4 these are not quite the expected old-age forms. The solutions of the linearized equation (5.4), in its original form, which match the above solutions of the linearized large R' version of (5.12) can be found as

$$V \sim \epsilon^{\frac{1}{2}} C_1(a) \exp\{-\tfrac{1}{4}\epsilon R^2 - \tfrac{1}{2}a\epsilon^{\frac{1}{2}}R + \tfrac{1}{2}\epsilon R\} \sin\theta \quad \text{if} \quad j = 1, \tag{5.15}$$

$$V \sim \epsilon C_2(a) \exp\{a \exp[\epsilon(R-1)/a]\} \sin\theta \quad \text{if} \quad j = 2, \tag{5.16}$$

for $R\epsilon^{\frac{1}{2}j} \gg 1$.

Returning to the physical variables, these read

$$U \sim [a_0^2/(\gamma+1)\, \omega (rr_0)^{\frac{1}{2}}]\, D_1(\alpha r_0) \exp(-\alpha r) \sin(\omega\tau) \quad \text{if} \quad j = 1, \tag{5.17}$$

$$U \sim [a_0^2/(\gamma+1)\, \omega r]\, D_2(\alpha r_0) \exp(-\alpha r) \sin(\omega\tau) \quad \text{if} \quad j = 2, \tag{5.18}$$

where D_1, D_2 are undetermined functions of the parameter αr_0, which is equivalent to a and is to be regarded as a *fixed* parameter here. The expressions (5.17, 5.18) show amplitude saturation as expected – and as confirmed by experiment (Shooter *et al.* 1974). Unfortunately they still contain the unknown functions D_1, D_2 of the single variable (αr_0), and for the determination of these functions it would appear that nothing short of the general solution to the Burgers equation for cylindrical and spherical waves will suffice. A heuristic argument given by Shooter *et al.* (1974) leads also to the spherical wave solution (5.18) provided that

$$D_2(\alpha r_0) = 4 \exp\{\Gamma(\alpha r_0)\}\, \Gamma(\alpha r_0), \tag{5.19}$$

where the function Γ is defined implicitly by

$$\Gamma(x) \ln\{\Gamma(x)/x\} = 1. \tag{5.20}$$

It is remarkable that their proposal has the structure (5.18) which emerges only after a fairly intricate asymptotic analysis, and in view of the fact that (5.19) gives agreement with the experiments of Shooter *et al.* on saturation of spherical waves in water it can be safely commended as adequate for all purposes except that of understanding the mechanics bound up in the intractable equation (5.12).

6. N-waves in a relaxing gas

We conclude with a brief discussion of the propagation of plane N-waves in a relaxing gas. The model equation will be taken in a form whose structure will be appreciated from the work of §§ 2–4, namely

$$\left(1-a\epsilon\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial T}+u\frac{\partial u}{\partial x}-\delta\frac{\partial^2 u}{\partial x^2}\right)=\epsilon\frac{\partial^2 u}{\partial x^2}, \tag{6.1}$$

with

$$u(x,1)\begin{cases}=x & \text{for} \quad |x|<1,\\ =0 & \text{for} \quad |x|>1.\end{cases} \tag{6.2}$$

In (6.1) ϵ is the ratio of relaxation time to the time L/a_0 characteristic of a wave of length L, a is a fixed positive parameter, $\delta=o(\epsilon^3)$ is a very small thermoviscous diffusion coefficient. Precise details can be ascertained from a comparison of (6.1) with § 2.

The outer loss-less solution is the same as that for N-waves in a thermoviscous fluid (§ 3 with $g\equiv 1$), while the inner shock solution has the same form as was discussed in § 4 for harmonic waves in a relaxing gas. When $aT^{-\frac{1}{2}}>2$ the shock at the head of the wave is partly dispersed, and must contain a Taylor subshock of the usual kind. To be specific, let

$$x^*=(x-T^{\frac{1}{2}})/\epsilon, \quad u(x^*,T,\epsilon)=(1+u_0^*)/2T^{\frac{1}{2}}+o(1); \tag{6.3}$$

then

$$a\ln(1-u_0^{*2})+T^{\frac{1}{2}}\ln[(1-u_0^*)/(1+u_0^*)]=x^*-A(T), \tag{6.4}$$

where an integral conservation law determines $A(T)$ as

$$A(T)=-T^{\frac{1}{2}}\ln T-2a(\ln 2-1)-A_0, \tag{6.5}$$

and $A_0=0$ when there is no subshock,

$$A_0=2T^{\frac{1}{2}}\{aT^{-\frac{1}{2}}-2-\ln(\tfrac{1}{2}aT^{-\frac{1}{2}})(1+\tfrac{1}{2}aT^{-\frac{1}{2}})\}, \tag{6.6}$$

when there is a subshock.

Thus if $a>2$ there is a subshock up until $T=(\frac{1}{2}a)^2$, when an 'embryo subshock' region is needed, much as described earlier. There is again a non-uniformity at large T, actually for $\ln T\sim\epsilon^{-1}$, but when $T\gg\epsilon^{-2}$ the solution to the present problem coincides, to leading order, with that for N-waves in a thermoviscous fluid. The large time non-uniformity is therefore covered by the work of § 3 for N-waves governed by Burgers's equation.

The embryo shock region is interesting. The required rescaling to describe the initial motion is

$$\left.\begin{aligned}\hat{x}=(x-1)/\epsilon, \quad \hat{T}=(T-1)/\epsilon,\\ u(\hat{x},\hat{T},\epsilon)=\hat{u}_0+o(1),\end{aligned}\right\} \tag{6.7}$$

and the equation for $\hat{u}_0$ is the full relaxing gas model equation (with thermoviscous terms suppressed), namely

$$\left(1-a\frac{\partial}{\partial\hat{x}}\right)\left(\frac{\partial\hat{u}_0}{\partial\hat{T}}+\hat{u}_0\frac{\partial\hat{u}_0}{\partial\hat{x}}\right)=\frac{\partial^2\hat{u}_0}{\partial\hat{x}^2}. \tag{6.8}$$

Embedded inside the embryo shock region, there is an embryo subshock region, defined by the scaling

$$\tilde{x} = [x-1-a^{-1}(T-1)]/\delta, \quad \tilde{T} = (T-1)/\delta, \tag{6.9}$$

and the expansion

$$u(\tilde{x}, \tilde{T}, \epsilon) = \tilde{u}_0(\tilde{x}, \tilde{T}) + o(1), \tag{6.10}$$

where

$$\frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{u}_0}{\partial \tilde{T}} + \tilde{u}_0 \frac{\partial \tilde{u}_0}{\partial \tilde{x}} - \frac{\partial^2 \tilde{u}_0}{\partial \tilde{x}^2}\right) = 0, \tag{6.11}$$

with matching to the embryo shock region expressed by

$$\left.\begin{array}{lll} \tilde{u}_0 \to 0 & \text{as} & \tilde{x} \to +\infty, \\ \tilde{u}_0 \to 1 & \text{as} & \tilde{x} \to -\infty. \end{array}\right\} \tag{6.12}$$

and

The solution in this embryo subshock region is found from the Cole–Hopf linearization as

$$\tilde{u}_0 = \left\{1 + \frac{\operatorname{erfc}\,(-\tilde{x}/2\tilde{T}^{\frac{1}{2}})\exp[-(\tilde{T}-2\tilde{x})/4]}{\operatorname{erfc}\,[(\tilde{x}-\tilde{T})/2\tilde{T}^{\frac{1}{2}}]}\right\}^{-1}. \tag{6.13}$$

As $\tilde{T} \to \infty$, $\tilde{u}_0$ goes over to the 'tanh' profile of the steady state Taylor subshock.

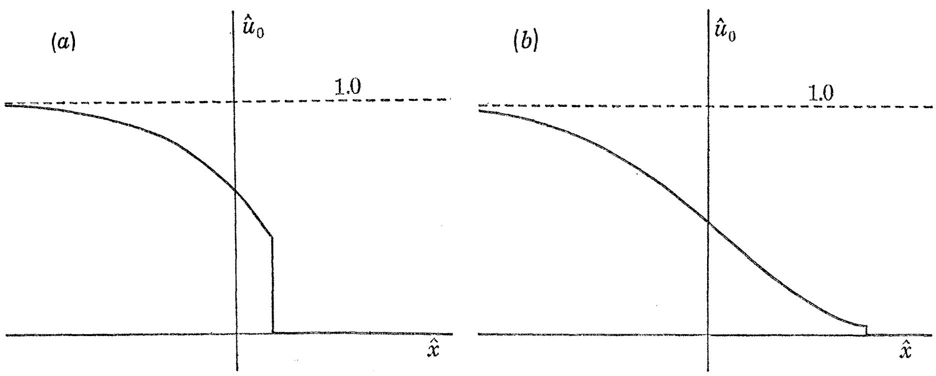

FIGURE 1. The waveform $\hat{u}_0(\hat{x})$, seen from a frame of reference moving with velocity $\frac{1}{2}$ at (a) $\hat{T} = O(1)$ and (b) $\hat{T} \gg 1$.

Even if $a \leqslant 2$, the subshock persists throughout the embryo shock region, and is located at $x_2(T, \epsilon)$ say. Thus the embryo shock solution u_0 above is zero for $x > x_2$, and satisfies the initial condition

$$\hat{u}_0(\hat{x}, 0) = 1 - H(\hat{x}), \tag{6.14}$$

where H is the Heaviside function. Equation (6.8) cannot, as yet, be solved exactly for $\hat{u}_0$, though the qualitative behaviour of $\hat{u}_0$ can be understood with the above conditions. For $\hat{T} \to \infty$, $\hat{u}_0$ goes over to the fully dispersed relaxing shock profile discussed in §4.

If $a > 2$, the 'step' persists into the fully developed relaxing shock in the form of a subshock, while if $a < 2$ the step amplitude becomes exponentially small outside the embryo shock region. In any case, it is evident how the step fits in to the solution after the embryo shock region; a sketch is given in figure 1 of the waveform $\hat{u}_0(\hat{x})$ at different times $\hat{T}$, with the subshock step in the front of the wave.

This completes a brief examination of the salient features of the head of the wave. For the situation at the other end of the wave a generally similar situation exists, though a simple reflexion argument is not adequate as the solution is not antisymmetric about $x = 0$.

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7. Conclusions

In this article we have attempted to show how the methods of matched asymptotic expansions can be applied in the study of nonlinear acoustics. Although the problems we have tackled are of interest in themselves they also serve as illustrations (sometimes with distinctly unusual features, as in § 3) of techniques with much wider applications. The process of solution starts with a formulation of the problem to be solved as an equation (or set of equations) with boundary and/or initial conditions. The equation may be an approximation itself or it may be exact, leaving aside the fact that few equations are strictly exact: even the Navier–Stokes equation is itself an approximation, neglecting many physical phenomena. The problem is then non-dimensionalized on some suitable scales, in general leaving several non-dimensional free parameters (Reynolds numbers, Mach numbers, etc.). One of these is assumed small and the remainder are to have *fixed* asymptotic relationships with this governing parameter; the choice is dictated by physical interests and attempts to make the resulting approximate problem soluble (see § 2, though note also that the choice dictated by physical requirements led to an intractable irreducible problem in § 5). An asymptotic expansion is assumed and the resulting approximate problems solved. In general there are places where this expansion breaks down and new regions must be introduced to cope with this. The method is essentially constructive – pointing out at each stage the need for a new region – although it may still be difficult to find the right stretchings or translations in the coordinates. Thus in § 4, for example, we were unable to find the correct region to describe the production of the subshock as the relaxing shock became partly dispersed; we feel that this is a question of coordinate translation but cannot justify this at present.

Often there are regions in which the problem will not simplify (i.e. the zeroth order problem is governed by the complete equation with no small parameters) or in which one cannot solve the zeroth order problem. In these cases results about the asymptotics of the zeroth order problem may still allow matching to other regions to be accomplished. Consider § 3, where we obtain the spherical old age solution by matching to an 'irreducible' region.

Even when the problem is not solvable in some regions, the asymptotic structure itself is of great interest. First, it often describes the physical components of the solution, shock waves, transition regions, etc., and in general this is more interesting than the detailed solutions obtained. Secondly, one may be able to prove results about the solution based on the scalings themselves, as, for example, in § 5, where we have shown that saturation occurs and have proved that the saturation amplitude has a particular form, leaving only one function of a single parameter undetermined. In this context we should mention the triple-deck schemes, used in high Reynolds number flows, in which the reduced problems are often not solvable analytically and where the emphasis is placed on the importance of asymptotic structure. The incredible accuracy of some of the results obtained by the triple-deck method is an added bonus.

In those regions where analytic solution is impossible one should prove the existence of a solution which matches the surrounding regions, for otherwise the choice of stretchings is in doubt. It is easy to construct examples of the incorrect choice of a region in which the solution will not match the other regions, and particularly if the stretchings are complicated and non-standard one cannot be sure they are right unless existence is proven. As an example, in § 3 the choice of regions for the transition of the spherical wave to old-age is not simple, and it is not obvious that there exists a solution to the resulting problem. An article is in preparation (Scott 1979) proving the existence result for this case, but a general theory of *existence with asymptotic boundary or initial conditions* needs to be developed.

Following on from this, it is less imperative, but nonetheless desirable to show uniqueness of solutions to the reduced equations with their corresponding matching conditions. Often the general solution to the zeroth order equation is known and in this case the matching conditions fix all unknown functions, thereby showing uniqueness. However, in some cases, e.g. the Fay solution and embryo shock region in § 4, the solution is known, but uniqueness is not guaranteed. Again a general theory of asymptotic uniqueness is required.

Other applications of the techniques illustrated in this paper will be made in articles now in preparation covering ionization fronts in H II regions, sound propagation in a stratified atmosphere and diffraction out of a sound beam of finite width. As can be seen, the methods do indeed cover a broad range of problems and deserve wider application than we have been able to suggest in this article.

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Appendix. Long time asymptotics of the solution of Burgers's equation for spherical waves

In § 3 we were led to the following problem:

$$\left.\begin{aligned} \frac{\partial V}{\partial T} + V\frac{\partial V}{\partial x} &= e^{T/T_0}\frac{\partial^2 V}{\partial x^2}, \\ V \sim -\tfrac{1}{2}\tanh\left(\tfrac{1}{4}x\,e^{-T/T_0}\right) \quad &\text{as} \quad T \to -\infty, \\ V \to \pm\tfrac{1}{2} \quad \text{as} \quad &x \to \pm\infty, \end{aligned}\right\} \tag{A 1}$$

which will be discussed in a forthcoming article (Scott 1979) from a pure mathematical standpoint. For the moment, however, we will only need to know that V is bounded, which seems physically evident.

Put $\nu = e^{T/T_0}/T_0$, to find $$\frac{\partial V}{\partial \nu} - T_0^2\frac{\partial^2 V}{\partial x^2} = -\frac{T_0}{\nu}V\frac{\partial V}{\partial x},$$

and, looking for the solution as $T \to \infty$, we let V_0 be the solution of

$$\frac{\partial V_0}{\partial \nu} = T_0^2\frac{\partial^2 V_0}{\partial x^2} \quad \text{with} \quad V_0(x, \nu = 1) = V(x, \nu = 1).$$

Then
$$\begin{aligned} V &= V_0 - \int_1^{\nu}\int_{-\infty}^{\infty} \frac{(VV_x)(x', \nu')\, e^{-(x-x')^2/4(\nu-\nu')T_0^2}\,dx'\,d\nu'}{\nu'[4\pi(\nu-\nu')]^{\frac{1}{2}}}, \\ &= V_0 - \int_1^{\nu}\int_{-\infty}^{\infty} \frac{(x'-x)\,V^2(x', \nu')\, e^{-(x-x')^2/4(\nu-\nu')T_0^2}\,dx'\,d\nu'}{T_0^2\,\nu'\sqrt{\pi}[4(\nu-\nu')]^{\frac{3}{2}}}, \\ &= V_0 - \int_1^{\nu}\int_{-\infty}^{\infty} \frac{y V^2(x + 2T_0(\nu-\nu')^{\frac{1}{2}}y, \nu')\, e^{-y^2}\,dy\,d\nu'}{\nu'\sqrt{\pi}[4(\nu-\nu')]^{\frac{1}{2}}}. \end{aligned}$$

Let K be a bound on V (i.e. $|V| \leqslant K$ for all x and ν). Then

$$|V-V_0| \leqslant \frac{K^2}{2\sqrt{\pi}}\int_1^{\nu}\int_{-\infty}^{\infty}\frac{|y|\,\mathrm{e}^{-y^2}\,\mathrm{d}y\,\mathrm{d}\nu'}{\nu'(\nu-\nu')^2},$$

$$= \frac{K^2}{2\sqrt{\pi}\nu^{\frac{1}{2}}}\ln\left(\frac{\nu^{\frac{1}{2}}+(\nu-1)^{\frac{1}{2}}}{\nu^{\frac{1}{2}}-(\nu-1)^{\frac{1}{2}}}\right)\to 0 \quad \text{as} \quad \nu\to\infty.$$

Therefore $V \sim V_0$ as $\nu\to\infty$.

The asymptotics of V as $T\to\infty$ are now reduced to those of V_0, a much easier problem since V_0 satisfies the diffusion equation. Letting

$$\alpha = V_0 - \tfrac{1}{2}[\operatorname{erfc}(x/2T_0\nu^{\frac{1}{2}})-1],$$

then α also satisfies the diffusion equation, and so if $\beta(x) = \alpha(x, \nu = 1)$ we have

$$\alpha = \int_{-\infty}^{\infty}\frac{\beta(x')\,\mathrm{e}^{-(x-x')^2/4(\nu-1)T_0^2}\,\mathrm{d}x'}{T_0[4\pi(\nu-1)]^{\frac{1}{2}}}.$$

Next we form an estimate of this integral; because $V\to\pm\frac{1}{2}$ as $x\to\pm\infty$, $\beta(x)\to 0$ at infinity and so $|\beta(x)| < \epsilon$ for $|x| > X(\epsilon)$, for any $\epsilon > 0$. By using the bound on V, $|\beta(x)| < K+\frac{1}{2}$ for all x. Therefore

$$|\alpha| \leqslant \frac{1}{T_0[4\pi(\nu-1)]^{\frac{1}{2}}}\left\{\int_{|x'|<X(\epsilon)}+\int_{|x'|>X(\epsilon)}|\beta(x')|\,\mathrm{e}^{-(x-x')^2/4(\nu-1)T_0^2}\,\mathrm{d}x'\right\},$$

$$< \frac{1}{T_0[4\pi(\nu-1)]^{\frac{1}{2}}}\{(K+\tfrac{1}{2})\,2X(\epsilon)+\epsilon T_0[4\pi(\nu-1)]^{\frac{1}{2}}\},$$

$$= \frac{(K+\frac{1}{2})\,X(\epsilon)}{T_0[\pi(\nu-1)]^{\frac{1}{2}}}+\epsilon,$$

so for given $\epsilon > 0$ we can find ν_0 sufficiently large that $|\alpha| < 2\epsilon$ for $\nu > \nu_0$, i.e. $\alpha\to 0$ as $\nu\to\infty$, uniformly in x. Our conclusion is that

$$V \sim V_0 \sim \tfrac{1}{2}(\operatorname{erfc}(x/2T_0^{\frac{1}{2}}\,\mathrm{e}^{T/2T_0})-1) \quad \text{as} \quad T\to\infty,$$

which is the statement made in §3 and which we have now justified.

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